Chiral Perturbation Theory & Scattering Amplitude Calculations

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Abstract

For the sake of further and better understanding of QFT, an internship launched in Hunan University, focusing on Chiral Perturbation Theory in order to calculate the scattering amplitude for processes. Herein, there are brief reviews of Peskin's "An Introduction to Quantum Field Theory", as well as Stefan Scherer and Matthias R. Schindler's "A Primer for Chiral Perturbation Theory". Then taken as the practice, two procedures $\omega \to \pi\gamma$ and $e^- e^+ \to \pi^+\pi^-\eta$ were treated by ChPT, for the scattering amplitude and form factors, respectively. Later, we are eager to figure out the state mixing for the second process.

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Part I
QFT - Peskin

Ref: "An Introduction to Quantum Field Theory" of Peskin

1 Fields and propagators

1.1 K-G propagators

In the Heisenberg picture, the amplitude for a particule to propagate from y to x is $\langle 0|\phi(x)\phi(y)|0\rangle$; we call this quantity $D(x-y)$.

$$\boxed{D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle}$$

Quantization of K-G field shown below:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{2.25}$$

$$\phi(x) = \phi(\mathbf{x},t) = e^{iHt}\phi(\mathbf{x})e^{-iHt} \tag{2.43}$$

$$\phi(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}}e^{+ip\cdot x}\right)\Big|_{p^0=E_{\mathbf{p}}} \tag{2.47}$$

Notice $\hat{a}^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}$ in eq. 2.25 but $\hat{a}^\dagger_{\mathbf{p}}e^{+ip\cdot x}$ in 2.47.

$$D(x-y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}$$

$D_R(x-y)$ is a Green's function of the K-G operator. Since it vanishes for $x^0 < y^0$, it is the retarded Green's function.

$$\langle 0|\left[\phi(x),\phi(y)\right]|0\rangle = D(x-y) - D(y-x)\begin{cases} = 0 \text{ spacelike} \\ \neq 0 \text{ timelike}\end{cases}$$

$$\begin{aligned}
\langle 0|\left[\phi(x),\phi(y)\right]|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)} - e^{+ip\cdot(x-y)}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\left\{\frac{1}{+2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}\Big|_{p^0=+E_{\mathbf{p}}} + \frac{1}{-2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}\Big|_{p^0=-E_{\mathbf{p}}}\right\} \\
&\overset{x^0>y^0}{=\!=} \int \frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip\cdot(x-y)}
\end{aligned}$$

$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|\left[\phi(x),\phi(y)\right]|0\rangle}$$

where $\theta(x^0 - y^0)$ is Heaviside function, whose derivative is Dirac function (distribution). By Fourier transform, we have:

$$(\partial^2 + m^2)D_R(x-y) = -i\delta^{(4)}(x-y) \qquad \Rightarrow \qquad (-p^2+m^2)\tilde{D}_R(p) = -i$$

$$D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|\,[\phi(x),\phi(y)]\,|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}e^{-ip\cdot(x-y)}$$

In order to avoid the singularity, we add $+i\epsilon$ on the denominator, obtaining $D_F(x-y)$, which is a Green's function of the K-G operator. The Green's function $D_F(x-y)$ is called the Feynman propagator for a K-G particles.

$$\boxed{D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip\cdot(x-y)}}$$

$$\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & for\ x^0 > y^0 \\ D(y-x) & for\ x^0 < y^0 \end{cases} \\ &= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &\equiv \langle 0|T\phi(x)\phi(y)|0\rangle \end{aligned}$$

"time-ordering" symbol T, instructs us to place the operators in order w/ the latest to the left.

1.2 Perturbation Expansion of Correlation Functions

- $|0\rangle$, ground state of the free theory;
- $|\Omega\rangle$, ground state of the interacting theory;
- $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$, two-point correlation function/ two-point Green's functions.

In the Heisenberg picture/field

$$\phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt} \tag{4.13}$$

$$\phi(t,\vec{x}) = e^{iH(t-t_0)}\phi(t_0,\vec{x})e^{-iH(t-t_0)}$$

Hamiltonian of ϕ^4 theory

$$H = H_0 + H_{\text{int}} = H_{\text{K-G}} + \int d^3x \frac{\lambda}{4!}\phi^4(\vec{x}) \tag{4.12}$$

for $\lambda = 0$, H becomes H_0

$$\boxed{\phi(t,\vec{x})|_{\lambda=0} = e^{iH_0(t-t_0)}\phi(t_0,\vec{x})e^{-iH_0(t-t_0)} \equiv \phi_I(t,\vec{x})} \tag{4.14}$$

when λ small, this expression will still give the most important part of the time dependence of $\phi(x)$, then $\phi_I(t,\vec{x})$ called the interaction picture field.

$$\phi_I(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\hat{a}_{\mathbf{p}}e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p}}e^{+ip\cdot x}\right)\Bigg|_{x^0=t-t_0} \tag{4.15}$$

* First, express ϕ by ϕ_I.

$$\begin{aligned}\phi(t,\vec{x}) &= e^{+iH(t-t_0)}\phi(t_0,\vec{x})e^{-iH(t-t_0)} \\ &\cong e^{+iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(t_0,\vec{x})e^{+iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &\equiv U^\dagger(t,t_0)\phi_I(t,\vec{x})U(t,t_0)\end{aligned} \tag{4.16}$$

where $U(t,t_0)$ is the interaction picture propagator time-evolution operator:

$$\begin{aligned}U(t,t_0) &= e^{+iH_0(t-t_0)}e^{-iH(t-t_0)} \\ U^\dagger(t,t_0) &= e^{+iH(t-t_0)}e^{-iH_0(t-t_0)}\end{aligned} \tag{4.17}$$

* Then calculate the derivative of $U(t,t_0)$ to derive it from ϕ_I

$$\begin{aligned}i\frac{\partial}{\partial t}U(t,t_0) &= e^{+iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)} \\ &= \left[e^{+iH_0(t-t_0)}(H_{\text{int}})e^{-iH_0(t-t_0)}\right]\cdot\left[e^{+iH_0(t-t_0)}e^{-iH(t-t_0)}\right] \\ &= H_I(t)\cdot U(t,t_0)\end{aligned} \tag{4.18}$$

$H_I(t)$ is Hamiltonian for interaction, as a function of ϕ_I:

$$H_I(t) = e^{+iH_0(t-t_0)}(H_{\text{int}})e^{-iH_0(t-t_0)} = \int d^3x\frac{\lambda}{4!}\phi_I^4 \tag{4.20}$$

Note we could not integrate eq. 4.18 directly, expanding that with series instead,

$$\begin{aligned}U(t,t_0) &= 1+(-i)\int_{t_0}^t dt_1H_I(t_1)+(-i)^2\int_{t_0}^t dt_1\int_{t_0}^{t_1} dt_2H_I(t_1)H_I(t_2) \\ &\quad +(-i)^3\int_{t_0}^t dt_1\int_{t_0}^{t_1} dt_2\int_{t_0}^{t_2} dt_3H_I(t_1)H_I(t_2)H_I(t_3)+\cdots \\ &= 1+(-i)\int_{t_0}^t dt_1H_I(t_1)+\frac{(-i)^2}{2!}\int_{t_0}^t dt_1\int_{t_0}^{t_1} dt_2T\left\{H_I(t_1)H_I(t_2)\right\}+\cdots \\ &\equiv T\left\{\exp\left[-i\int_{t_0}^t dt'H_I(t')\right]\right\}\end{aligned} \tag{4.22}$$

* Next derive $|\Omega\rangle$ (ground state of the interacting theory) from $|0\rangle$(ground state of the free theory). Suppose $\langle\Omega|0\rangle\neq 0$, E_n is eigenvalues of H, $E_0\equiv\langle\Omega|H|\Omega\rangle$

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle$$

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle+\sum_{n\neq 0}e^{-iE_nT}|n\rangle\langle n|0\rangle \simeq e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$$

$$\begin{aligned}|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)}\left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1}e^{-iHT}|0\rangle \\ &\simeq \lim_{T\to\infty(1-i\epsilon)}\left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1}e^{-iH(T+t_0)}|0\rangle \\ &= \lim_{T\to\infty(1-i\epsilon)}\left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1}e^{-iH(t_0-(-T))}e^{-iH_0(-T-t_0)}|0\rangle \\ &= \lim_{T\to\infty(1-i\epsilon)}\left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1}U(t_0,-T)|0\rangle\end{aligned} \tag{4.28}$$

Notice there is an approximation $T + t_0 \approx T$ on the second row. Since $\hat{H}_0|0\rangle = 0$, we add $e^{-iH_0(-T-t_0)}$ in the third row. Similarly, we could write $\langle\Omega|$

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \tag{4.29}$$

According to normalization, $\langle\Omega|\Omega\rangle = \left[|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right]^{-1}\langle 0|U(T,t_0)U(t_0,-T)|0\rangle = 1$. Finally, we could get $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle$

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} &\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}\langle 0|U(T,t_0)\times\left[U(x^0,t_0)\right]^\dagger\phi_I(x)U(x^0,t_0) \\ &\times\left[U(y^0,t_0)\right]^\dagger\phi_I(y)U(y^0,t_0)\times U(t_0,-T)\left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1}\end{aligned} \tag{4.30}$$

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)}\frac{\langle 0|U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y_0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle} \\ &= \lim_{T\to\infty(1-i\epsilon)}\frac{\left\langle 0\left|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}\right|0\right\rangle}{\left\langle 0\left|T\left\{\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}\right|0\right\rangle}\end{aligned} \tag{4.31}$$

1.3 Wick's Theorem

Now divide the interaction term into two parts, $\phi_I^+(x)$ with generation operator and $\phi_I^-(x)$ with annihilation operator.

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\hat{a}_\mathbf{p}e^{-ip\cdot x} + \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\hat{a}_\mathbf{p}^\dagger e^{+ip\cdot x} \tag{4.32}$$

Obviously, $\phi^+|0\rangle = 0$, as well as $\langle 0|\phi^- = 0$. Consider two fields under time-ordering symbol T:

$$\begin{aligned}T\{\phi_I(x)\phi_I(y)\} &\xlongequal{x^0>y^0} \phi^+(x)\phi^+(y) + \left\{\phi^+(x)\phi^-(y)\right\} + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\ &= \phi^+(x)\phi^+(y) + \left\{\phi^-(y)\phi^+(x) + \left[\phi^+(x),\phi^-(y)\right]\right\} + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y)\end{aligned}$$

Define the contraction of two fields, it just equals to Feynman propagator

$$\overbrace{\phi(x)\phi}(y) = \begin{cases}\left[\phi^+(x),\phi^-(y)\right] & x^0 > y^0 \\ \left[\phi^+(y),\phi^-(x)\right] & x^0 < y^0\end{cases} = D_F(x-y)$$

Define the normal-ordering symbol $NN(\hat{a}_\mathbf{p}\hat{a}_\mathbf{k}^\dagger\hat{a}_\mathbf{q}) = \hat{a}_\mathbf{k}^\dagger\hat{a}_\mathbf{p}\hat{a}_\mathbf{q}$. Note any initial or final state treated by such a result would vanish! Therefore, we find the relation between time-ordering symbol T and contractions, which is Wick's Theorem.

$$T\{\phi(x_1)\phi(x_2)\cdots\phi(x_m)\} = N\{\phi(x_1)\phi(x_2)\cdots\phi(x_m) + \text{all possible contractions}\}$$

Consider four fields under time-ordering symbol for example:

$$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3)$$

As for ϕ^4 theory, we have to introduct four ϕ in one-order perturbation. According to the fundamentals of permutation, the number of fully contracted terms is equal to $(4+1)!! = 15$, $(8+1)!! = 945$, $(12+1)!! = 135135$ for order 1, 2, 3 respectively.

1.4 Dirac propagators

Review Dirac fields and related equation

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma \cdot p + m = \not{p} + m \tag{3.66}$$

$$\sum_s v^s(p)\bar{v}^s(p) = \gamma \cdot p - m = \not{p} - m \tag{3.67}$$

Here is Quantization of the Dirac field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(\hat{a}^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + \hat{b}^{s\,\dagger}_{\mathbf{p}} v^s(p) e^{+ip\cdot x} \right) \tag{3.99}$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(\hat{b}^s_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} + \hat{a}^{s\,\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{+ip\cdot x} \right) \tag{3.100}$$

Propagation amplitude of Dirac field:

$$\begin{aligned} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s u^s_a(p)\bar{u}^s_b(p) e^{-ip\cdot(x-y)} \\ &= (i\not{\partial}_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \end{aligned} \tag{3.114}$$

$$\begin{aligned} \langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s v^s_a(p)\bar{v}^s_b(p) e^{+ip\cdot(x-y)} \\ &= -(i\not{\partial}_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(y-x)} \end{aligned} \tag{3.115}$$

Define Retarded Green's function:

$$S^{ab}_R(x-y) \equiv \theta(x^0 - y^0)\langle 0| \left\{ \psi_a(x), \bar{\psi}_b(y) \right\} |0\rangle \tag{3.116}$$

It's easy to validate that

$$S_R(x-y) = (i\not{\partial}_x + m)D_R(x-y) = (i\not{\partial}_x + m)\theta(x^0 - y^0)\langle 0| \left[\phi(x)\phi(y)\right] |0\rangle \tag{3.117}$$

S_R is a Green's function of the Dirac operator

$$(i\not{\partial}_x - m)S_R(x-y) = i\delta^{(4)}(x-y) \cdot \mathbf{1}_{4\times 4} \tag{3.118}$$

$$i\delta^{(4)}(x-y) = \int \frac{d^4p}{(2\pi)^4} (\not{p} - m) e^{-ip\cdot(x-y)} \tilde{S}_R(p) \tag{3.119}$$

$$\tilde{S}_R(p) = \frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p^2 - m^2} \tag{3.120}$$

Also, we add $i\epsilon$ again.

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)} = \begin{cases} +\langle 0|\psi(x)\bar{\psi}(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle & x^0 < y^0 \end{cases} \tag{3.121}$$

Next, apply time-ordering symbol on Dirac fields:

$$T\left[\psi(x)\bar{\psi}(y)\right] \equiv \begin{cases} +\psi(x)\bar{\psi}(y) & x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) & x^0 < y^0 \end{cases} \tag{4.105}$$

$$S_F(x-y) = \langle 0|T\left[\psi(x)\bar{\psi}(y)\right]|0\rangle \tag{4.106}$$

Note that we should multiply -1. For example, if $x_3^0 > x_1^0 > x_4^0 > x_2^0$, we change positions three times (1,2)(1,3)(3,4) :

$$T(\psi_1\psi_2\psi_3\psi_4) = (-1)^3\psi_3\psi_1\psi_4\psi_2$$

Similar property for normal-ordering symbol N

$$N(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{r}}) = (-1)^2\hat{a}^\dagger_{\mathbf{r}}\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}} = (-1)^3\hat{a}^\dagger_{\mathbf{r}}\hat{a}_{\mathbf{q}}\hat{a}_{\mathbf{p}}$$

Define contraction, where $\bar{\psi} = \psi^\dagger\gamma^0$

$$T\left[\psi(x)\bar{\psi}(y)\right] = N\left[\psi(x)\bar{\psi}(y)\right] + \contraction{}{\psi}{(x)}{\bar{\psi}}\psi(x)\bar{\psi}(y) \tag{4.107}$$

$$\contraction{}{\psi}{(x)}{\bar{\psi}}\psi(x)\bar{\psi}(y) \equiv \begin{cases} +\left\{\psi^+(x), \bar{\psi}^-(y)\right\} & x^0 > y^0 \\ -\left\{\bar{\psi}^+(y), \psi^-(x)\right\} & x^0 < y^0 \end{cases} = S_F(x-y) \tag{4.108}$$

$$\contraction{}{\psi}{(x)}{\psi}\psi(x)\psi(y) = \contraction{}{\bar{\psi}}{(x)}{\bar{\psi}}\bar{\psi}(x)\bar{\psi}(y) = 0 \tag{4.109}$$

Define contractions under the normal-ordering symbol N, to include minus signs for operator interchanges:

$$N(\contraction{}{\psi}{_1\psi_2}{\bar{\psi}}\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4) = -\contraction{}{\psi}{_1}{\bar{\psi}}\psi_1\bar{\psi}_3 N(\psi_2\bar{\psi}_4) = -S_F(x_1-x_3)N(\psi_2\bar{\psi}_4)$$

With these conventions, Wick's Theorem takes the same form as before:

$$T(\psi_1\bar{\psi}_2\psi_3\cdots) = N(\psi_1\bar{\psi}_2\psi_3\cdots + \text{all possible contractions})$$

2 Feynman rules

2.1 Notions

2.1.1 Cross section σ, Decay rate Γ

Rappel: the generalization to higher correlation functions

$$\langle\Omega|T\left[\phi(x_1)\cdots\phi(x_n)\right]|\Omega\rangle = \begin{pmatrix} \text{sum of all connected diagrams} \\ \text{with } n \text{ external points} \end{pmatrix} \tag{4.57}$$

Cross section σ is just the total number of events divided by all of these quantities, which has units of area.

$$\sigma \equiv \frac{\text{Number of scattering events}}{\text{Number of partcules}} \tag{4.59}$$

To get the momenta of final-state particles, we have to specify the exact momenta desired, making σ infinitesimal. Define differential cross section $d\sigma/(d^3p_1 \cdots d^3p_n)$, or $d\sigma/d\Omega$ as usual, where Ω is volume element of momenta space.

The decay rate Γ of an unstable particle into a specified final state is defined as

$$\Gamma \equiv \frac{\text{Number of decays per unit time}}{\text{Number of the particule chosen}} \tag{4.62}$$

In nonrelativistic quantum mechanics, an unstable atomic state show up in scattering experiments as a resonance. Near the resonance energy E_0, the scattering amplitude is given by the Breit-Wigner formula:

$$f(E) \propto \frac{1}{E - E_0 + i\Gamma/2} \tag{4.63}$$

therefore the cross section has a peak of the form

$$\sigma \propto \frac{1}{(E - E_0)^2 + \Gamma^2/4}$$

The width of the resonance peak is equal to the decay rate of the unstable state.

In relativistic quantum mechanics, we have

$$\frac{1}{p^2 - m^2 + im\Gamma} \approx \frac{1}{2E_{\mathbf{p}}\left[p^0 - E_{\mathbf{p}} + i(m/E_{\mathbf{p}})\Gamma/2\right]} \tag{4.64}$$

the decay rate of the unstable particle in a general fram is $(m/E_{\mathbf{p}})\Gamma$.

2.1.2 S-matrix $\mathcal{M}$

Consider a process from two particles to more

$$\begin{aligned} {}_{\text{out}}\langle \mathbf{p}_1\mathbf{p}_2 \cdots | \mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}} \rangle_{\text{in}} &= \lim_{T\to\infty} \langle \underbrace{\mathbf{p}_1\mathbf{p}_2 \cdots}_{T} | \underbrace{\mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}}}_{-T} \rangle \\ &= \lim_{T\to\infty} \left\langle \mathbf{p}_1\mathbf{p}_2 \cdots \left| e^{-iH(2T)} \right| \mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}} \right\rangle \end{aligned} \tag{4.70}$$

The *in* and *out* states are related by the limit of a sequence of unitary operators. This limiting unitary operator is called the S-matrix. $S = \mathbf{1} + iT$, T-matrix contains interactions.

$${}_{\text{out}}\langle \mathbf{p}_1\mathbf{p}_2 \cdots | \mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}} \rangle_{\text{in}} \equiv \langle \mathbf{p}_1\mathbf{p}_2 \cdots |S| \, \mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}} \rangle \tag{4.71}$$

Note that the matrix elements of S should reflect 4-momentum conservation. Therefore, S or T should always contain a factor $\delta^{(4)}(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f)$. Define the invariant matrix element $\mathcal{M}$, namely scattering amplitude, $\mathcal{M}_{if} = \langle \text{out}|\text{in}\rangle = \langle \text{final}|\text{initial}\rangle$

$$\langle \mathbf{p}_1\mathbf{p}_2 \cdots |iT| \, \mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}} \rangle = (2\pi)^4 \delta^{(4)}(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f) \cdot i\mathcal{M}(k_{\mathcal{A}}, k_{\mathcal{B}} \to p_f) \tag{4.73}$$

Note of course all 4-momenta are on mass-shell: $p^0 = E_{\mathbf{p}}, k^0 = E_{\mathbf{k}}$

on/off-shellhttps://en.wikipedia.org/wiki/On_shell_and_off_shell
In physics, particularly in quantum field theory, configurations of a physical system that satisfy classical equations of motion are called *on the mass shell* or simply more often on shell, satisfying $E^2 - |\vec{p}|^2 = m^2$; while those that do not are called *off the mass shell*, or off shell.

Virtual particles corresponding to internal propagators in a Feynman diagram are in general allowed to be off shell, but the amplitude for the process will diminish depending on how far off shell they are. This is because the q^2-dependence of the propagator is determined by the 4-momenta of the incoming and outgoing particles. The propagator typically has singularities on the mass shell.

$\mathcal{M}$ could be derived from T-matrix or S-matrix.

$$\mathcal{P}(\mathcal{AB} \to 12 \cdots n) = \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \left| {}_{\text{out}} \langle \mathbf{p}_1 \cdots \mathbf{p}_n | \phi_A \phi_B \rangle_{\text{in}} \right|^2 \tag{4.74}$$

$$\begin{aligned} d\sigma =& \frac{1}{2E_{\mathcal{A}} 2E_{\mathcal{B}} \left| v_{\mathcal{A}} - v_{\mathcal{B}} \right|} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \\ & \times \left| \mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)}(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum p_f) \end{aligned} \tag{4.79}$$

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{CM}} = \frac{1}{2E_{\mathcal{A}} 2E_{\mathcal{B}} \left| v_{\mathcal{A}} - v_{\mathcal{B}} \right|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{\text{cm}}} \left| \mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{B}} \to p_1, p_2) \right|^2 \tag{4.84}$$

$$\sigma = \int |\mathcal{M}|^2 \, d\Omega$$

The definition of Γ assumes that the decaying particle is at rest, so the normalization factor $(2E_{\mathcal{A}})^{-1}$ becomes $(2m_{\mathcal{A}})^{-1}$. Thus the decay rate formula

$$d\Gamma = \frac{1}{2m_{\mathcal{A}}} \left(\prod_f \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f} \right) \left| \mathcal{M}(m_{\mathcal{A}} \to \{p_f\}) \right|^2 (2\pi)^4 \delta^{(4)}(p_{\mathcal{A}} - \sum p_f) \tag{4.86}$$

with proof in section 7.3.

$$\Gamma = \frac{1}{2m} \sum_f d \prod_f \left| \mathcal{M}(p \to f) \right|^2 \tag{7.63}$$

2.1.3 Why Trace

Here is a brief summary about why we have to calculate Trace of matrices while applying Feynman rules on $\mathcal{M}$. Take Dirac fields as an example

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \zeta \\ \sqrt{p \cdot \bar{\sigma}} \zeta \end{pmatrix} \tag{3.50}$$

with the corresponding conjugate transposition

$$u^\dagger(p) = \left(\zeta^\dagger\sqrt{p\cdot\sigma}, \zeta^\dagger\sqrt{p\cdot\bar{\sigma}}\right)$$

and one scalar obtained by inner product

$$u^\dagger u = \left(\zeta^\dagger\sqrt{p\cdot\sigma}, \zeta^\dagger\sqrt{p\cdot\bar{\sigma}}\right)\cdot\begin{pmatrix}\sqrt{p\cdot\sigma}\zeta\\ \sqrt{p\cdot\bar{\sigma}}\zeta\end{pmatrix} = \zeta^\dagger\sqrt{p\cdot\sigma}\cdot\sqrt{p\cdot\sigma}\zeta+\zeta^\dagger\sqrt{p\cdot\bar{\sigma}}\cdot\sqrt{p\cdot\bar{\sigma}}\zeta = 2E_{\mathbf{p}}\zeta^\dagger\zeta \quad (3.55)$$

Change the order of multiplication above. Note that for the same spin $\zeta\zeta^\dagger = \mathbf{1}$.

$$\begin{aligned} uu^\dagger &= \begin{pmatrix}\sqrt{p\cdot\sigma}\zeta\\ \sqrt{p\cdot\bar{\sigma}}\zeta\end{pmatrix}\cdot\left(\zeta^\dagger\sqrt{p\cdot\sigma}, \zeta^\dagger\sqrt{p\cdot\bar{\sigma}}\right) = \begin{pmatrix}\sqrt{p\cdot\sigma}\zeta\cdot\zeta^\dagger\sqrt{p\cdot\sigma} & \sqrt{p\cdot\bar{\sigma}}\zeta\cdot\zeta^\dagger\sqrt{p\cdot\bar{\sigma}}\\ \sqrt{p\cdot\sigma}\zeta\cdot\zeta^\dagger\sqrt{p\cdot\sigma} & \sqrt{p\cdot\bar{\sigma}}\zeta\cdot\zeta^\dagger\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}}\\ \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}}\end{pmatrix}\end{aligned}$$

the trace just equals to the previous inner product.

$$\mathrm{Tr}(uu^\dagger) = \mathrm{Tr}\begin{pmatrix}\sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}}\\ \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}}\end{pmatrix} = \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma}+\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} = u^\dagger u$$

Moreover, take an instance of Peskin on page 132, $e^-e^+\to\mu^-\mu^+$

$$i\mathcal{M}[e^-(p)e^+(p')\to\mu^-(k)\mu^+(k')] = \frac{ie^2}{q^2}\left[\bar{v}(p')\gamma^\mu u(p)\right]\left[\bar{u}(k)\gamma_\mu v(k')\right] \quad (5.1)$$

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left[\bar{v}(p')\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v(p')\right]\left[\bar{u}(k)\gamma_\mu v(k')\bar{v}(k')\gamma_\nu u(k)\right] \quad (5.2)$$

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p}+m \qquad \sum_s v^s(p)\bar{v}^s(p) = \not{p}-m \quad (5.3)$$

$$\sum_{s,s'}\bar{v}_a^{s'}(p')\gamma^\mu_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^\nu_{cd}v^{s'}_d(p') = (\not{p}'-m)_{da}\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd} = \mathrm{Tr}\left[(\not{p}'-m)\gamma^\mu(\not{p}+m)\gamma^\nu\right]$$

Column vectors: u,v, row vectors by adding bar; gamma: Dirac matrices; abcd indexes for multiplication. We just saw the second = above; pay attention to the first =, suppose $A = \bar{v}_a^{s'}(p')\gamma^\mu_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^\nu_{cd}v^{s'}_d(p')$

$$\begin{aligned}\left[v^{s'}_d(p')\right]\cdot\bar{v}_a^{s'}(p')\gamma^\mu_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^\nu_{cd}v^{s'}_d(p')\cdot\left[v^{s'}_d(p')\right]^{-1} &= \left[v^{s'}_d(p')\right]\cdot A\cdot\left[v^{s'}_d(p')\right]^{-1}\\ = A\cdot\left[v^{s'}_d(p')\right]\cdot\left[v^{s'}_d(p')\right]^{-1} = A&\\ = \bar{v}_a^{s'}(p')\gamma^\mu_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^\nu_{cd}v^{s'}_d(p') = (\not{p}'-m)_{da}\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}&\end{aligned}$$

2.2 Magnetic vector potential & QED Feynman rules

Now we are ready to step from Yukawa theory to QED. To do this, we replace the scalar particle ϕ with a vector particle A_μ, and replace the Yukawa interaction Hamiltonian with

$$H_{\text{int}} = \int d^3x e\bar{\psi}\gamma^\mu\psi\mathscr{A}_\mu$$

with Feynman rules:

New vertex: $-ie\gamma^\mu$ Photon propagator: $\frac{-ig_{\mu\nu}}{q^2+i\epsilon}$ External photon lines: $\epsilon_\mu(p)$ or $\epsilon^*_\mu(p)$

Photons are conventionally drawn as wavy lines! The symbol $\epsilon_\mu(p)$ stand for the polarization vector of the initial- of final-state photon. To justify these rules, recall that in Lorentz gauge the field eq. for A_μ is $\partial^2\mathscr{A}_\mu = 0$, Thus each component of A separately obeys the Klein-Gordon eq. (w/ $m = 0$). The momentum-space solutions of this equation are $\epsilon_\mu(p)e^{-ip\cdot x}$ where $p^2 = 0$ and $\epsilon_\mu(p)$ is any 4-vector. The interpretation of ϵ as the polarization vector of the field should be familiar from classical electromagnetism. If we expand the quantized electromagnetic field in terms of classical solutions of the wave equation, as we did for the Klein-Gordon field, we find

$$\boxed{\mathscr{A}_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{r=0}^{3}\left(a^r_{\mathbf{p}}\epsilon^r_\mu(p)e^{-ip\cdot x} + a^{r\dagger}_{\mathbf{p}}\epsilon^{r*}_\mu(p)e^{ip\cdot x}\right)} \tag{4.131}$$

where $r = 0, 1, 2, 3$ labels a basis of polarization vectors. The external line factors in the Feynman rules above follow immediately from this expansion, just as we obtained u's and v's as the external line factors for Dirac particles. The only subtlety is that we must restrict initial- and final-state photons to be transversely polarized: Their polarization vectors are always of the form $\epsilon^\mu = (0, \boldsymbol{\epsilon})$, where $\mathbf{p}\cdot\boldsymbol{\epsilon} = 0$. For $\mathbf{p}$ along the z-axis, the right- and left-handed polarization vectors are $\epsilon^\mu = (0, 1, \pm i, 0)/\sqrt{2}$.

Define Mandelstam variables for 2-body → 2-body processes

$$s = (p_a + p_b)^2 \qquad t = (p_a - p_c)^2 \qquad u = (p_a - p_d)^2$$

According to 4-momenta conservation, $s + t + u = p_a^2 + p_b^2 + p_c^2 + p_d^2$. In general,

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

2.3 Photon polarization vector

Ref: Peskin page 159, 5.5 Compton Scattering, Photon Polarization Sums

$$\sum_{\text{polarizations}} \epsilon^*_\mu\epsilon_\nu \to -g_{\mu\nu} \tag{5.75}$$

This arrow indicates that this is not an actual equality. Nevertheless, the replacement is valide as long as both sides are dotted into the rest of the expression for a QED amplitude $\mathcal{M}$.

To derive this formula, let's consider an arbitrary QED process involving an external photon with momentum k: $i\mathcal{M}(k) \equiv i\mathcal{M}^\mu(k)\epsilon^*_\mu(k)$. Since the amplitude always contains $\epsilon^*_\mu(k)$, we have extracted this factor and defined $\mathcal{M}^\mu(k)$ to be the rest of the amplitude $\mathcal{M}$. The cross section will be proportional to

$$\sum_\epsilon \left|\epsilon^*_\mu(k)\mathcal{M}^\mu(k)\right|^2 = \sum_\epsilon \epsilon^*_\mu \epsilon_\nu \mathcal{M}^\mu(k)\mathcal{M}^{\nu *}(k)$$

For simplicity, we orient k in the 3-direction: $k^\mu = (k, 0, 0, k)$. Then the two transverse polarization vectors, over which we are summing, can be chosen be

$$\epsilon^\mu_1 = (0, 1, 0, 0) \qquad \epsilon^\mu_2 = (0, 0, 1, 0)$$

With these conventions, we have

$$\sum_\epsilon \left|\epsilon^*_\mu(k)\mathcal{M}^\mu(k)\right|^2 = \left|\mathcal{M}^1(k)\right|^2 + \left|\mathcal{M}^2(k)\right|^2 \tag{5.77}$$

Now recall that external photons are created by the interaction term $\int d^4x e j^\mu A_\mu$, where $j^\mu = \bar\psi\gamma^\mu\psi$ is the Dirac vector current. Therefore, we expect $\mathcal{M}^\mu(k)$ to be given by a matrix element of the Heisenberg field j^μ:

$$\mathcal{M}^\mu(k) = \int d^4x e^{ik\cdot x}\langle \text{final states}|j^\mu(x)|\text{initial states}\rangle \tag{5.78}$$

where the initial and final states include all particles except the photon in question. ... the current j^μ is conserved: $\partial_\mu j^\mu(x) = 0$. Provided that this property still holds in the quantum theory, we can dot k_μ into (5.78) to obtain

$$k_\mu \mathcal{M}^\mu(k) = 0 \tag{5.79}$$

The amplitude $\mathcal{M}$ vanishes when the polarization vector $\epsilon_\mu(k)$ is replaced by k_μ. This famous relation is know as the Ward identity. For $k^\mu = (k, 0, 0, k)$, the Ward identity takes the form:

$$k\mathcal{M}^0(k) - k\mathcal{M}^3(k) = 0 \tag{5.80}$$

Thus $\mathcal{M}^0 = \mathcal{M}^3$, and we have

$$\begin{aligned}\sum_\epsilon \epsilon^*_\mu \epsilon_\nu \mathcal{M}^\mu(k)\mathcal{M}^{\nu *}(k) &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 \\ &= |\mathcal{M}^1|^2 + |\mathcal{M}^2|^2 + |\mathcal{M}^3|^2 - |\mathcal{M}^0|^2 \\ &= -g_{\mu\nu}\mathcal{M}^\mu(k)\mathcal{M}^{\nu *}(k)\end{aligned}$$

That is, we may sum over external photon polarizations by replacing $\sum \epsilon^*_\mu \epsilon_\nu$ with $-g_{\mu\nu}$.

3 Renormalization

3.1 Path Integral

In a given time (T), one particle travels from x_a to x_b, we call this amplitude $U(x_a, x_b; T)$, given by in the canonical Hamiltonian formalism

$$U(x_a, x_b; T) = \left\langle x_b \left| e^{-iHT/\hbar} \right| x_a \right\rangle \tag{9.1}$$

There would be infinite paths between two points, we might therefore write the total amplitude

$$U(x_a, x_b; T) = \sum_{\text{all paths}} e^{i\cdot(\text{phase})} = \int \mathcal{D}x(t) e^{i\cdot(\text{phase})} \tag{9.2}$$

To be democratic, we have written the amplitude for each particular path as a pure phase, so that no path is inherently more important than any other. $\int \mathcal{D}x(t)$ is simply another way of writing "sum over all paths". Combine two formulas and the propagation amplitude is thus

$$\left\langle x_b \left| e^{-iHT/\hbar} \right| x_a \right\rangle = U(x_a, x_b; T) = \int \mathcal{D}x(t) e^{iS[x(t)]/\hbar} \tag{9.3}$$

S the discretized path, denoted by k as intermediate points

$$S = \int_0^T dt \left(\frac{m}{2}\dot{x}^2 - V(x) \right) \to \sum_k \left[\frac{m}{2} \frac{(x_{k+1} - x_k)^2}{\epsilon} - \epsilon V \left(\frac{x_{k+1} + x_k}{2} \right) \right]$$

We then define the path integral by ($C(\epsilon) = \sqrt{\frac{2\pi\hbar\epsilon}{-im}}$ is constant)

$$\int \mathcal{D}x(t) \equiv \frac{1}{C(\epsilon)} \int \frac{dx_1}{C(\epsilon)} \int \frac{dx_2}{C(\epsilon)} \cdots \int \frac{dx_{N-1}}{C(\epsilon)} = \frac{1}{C(\epsilon)} \prod_k \int_{-\infty}^{\infty} \frac{dx_k}{C(\epsilon)} \tag{9.4}$$

3.1.1 Functional Quantization of Scalar Fields

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right]$$

$$\left\langle \phi_b(\mathbf{x}) \left| e^{-iHT} \right| \phi_a(\mathbf{x}) \right\rangle = \int \mathcal{D}\phi \mathcal{D}\pi \exp \left[i \int_0^T d^4x \left(\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right) \right]$$

where the functions $\phi(x)$ over which we integrate are constrained to the specific configurations $x^0 = 0, x^0 = T$. Evaluate the $\mathcal{D}\pi$ integral to obtain

$$\left\langle \phi_b(\mathbf{x}) \left| e^{-iHT} \right| \phi_a(\mathbf{x}) \right\rangle = \int \mathcal{D}\phi \exp \left[i \int_0^T d^4x \mathcal{L} \right] \tag{9.14}$$

where $\mathcal{L} = (\partial_\mu \phi)^2/2 - V(\phi)$.

$$\int \mathcal{D}\phi e^{iS_0} = \text{constant} \times \left[\det(m^2 + \partial^2) \right]^{-1/2} \tag{9.25}$$

3.2 Effective Action

Ref: Peskin Chapitre 11, The Effective Action as a Generating Functional

$Z[J]$ is the generating functional of correlation functions, with respect to $J(x)$ produce the correlation functions of the scalar field (see eq 9.35). ... suppose $E[J] = i\log(Z[J])$, according to eq. 11.44

$$\begin{aligned}\frac{\delta^2 E[J]}{\delta J(x)\delta J(y)} &= -\frac{i}{Z}\int \mathcal{D}\phi e^{i\int(\mathcal{L}+J\phi)}\phi(x)\phi(y) + \frac{i}{Z^2}\int \mathcal{D}\phi e^{i\int(\mathcal{L}+J\phi)}\phi(x)\cdot\int \mathcal{D}\phi e^{i\int(\mathcal{L}+J\phi)}\phi(y) \\ &= -i\left[\langle\phi(x)\phi(y)\rangle - \langle\phi(x)\rangle\langle\phi(y)\rangle\right]\end{aligned} \tag{11.82}$$

emmmmmmmm

$$D(x,y) = \int \frac{d^4p}{(2\pi)^2} e^{-ip\cdot(x-y)}\tilde{D}(p) \tag{11.91}$$

We showed in eq 7.43 that momentum space propagator $\tilde{D}(p)$ is a geometric series in one-particle-irreducible Feynman diagrams. The Fourier transform of $D^{-1}(x,y)$ then gives the inverse propagator:

$$\tilde{D}^{-1}(p) = -i\left[p^2 - m^2 - M^2(p^2)\right] \tag{11.92}$$

where $M^2(p^2)$ is the sum of one-particle-irreducible two-point diagrams.

4 Non-Abelian gauge theory

Ref: Peskin Chapitre 15

4.1 The Geometry of Gauge Invariance

Consider complex-valued Dirac fields $\psi(x)$, and stipulate that our theory should be invariant under the transformation where $\alpha(x)$ could take any value.

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \tag{15.1}$$

Obviously, $m\bar{\psi}\psi$ keeps invariant as a scalar, permitted by global symmetry, and the local invariance gives no further restriction. The derivative of $\psi(x)$ in the direction of the vector n^ν is defined by the limiting procedure

$$n^\mu\partial_\mu\psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\psi(x+\epsilon n) - \psi(x)\right] \tag{15.2}$$

However, such a definition could not be applied in a theory with local phase invariance. Define a scalar quantity $U(y,x)$ that depends on the two points and has the transformation law

$$U(y,x) \to e^{i\alpha(y)}U(y,x)e^{-i\alpha(x)} \tag{15.3}$$

At zero separation, we set $U(y,y) = 1$ and $U(y,x) = \exp\left[i\phi(y,x)\right]$, so that $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation law. Define covariant derivative:

$$n^\mu D_\mu\psi = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\right] \tag{15.4}$$

Suppose $U(y,x)$ is a continuous function between y and x, then $U(y,x)$ can be expanded in the separation of two points:

$$U(x+\epsilon n, x) = 1 - ie\epsilon n^\mu A_\mu(x) + \mathcal{O}(\epsilon^2) \tag{15.5}$$

Here we have arbitrarily extracted a constant e, namely electron charge unit. The coefficient of the displacement ϵn^μ is a new vector field $A_\mu(x)$, namely magnetic vector potential. Such a field, which appears as the infinitesimal limit of a comparator of local symmetry transformations, is called a *connection.* The covariant derivative then takes the form

$$D_\mu \psi(x) = \partial_\mu \psi(x) + ieA_\mu \psi(x) \tag{15.6}$$

By inserting 15.5 into 15.3, we find that $A_\mu(x)$ transforms under this local gauge transformation as

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x) \tag{15.7}$$

Thus the covariant derivative transforms in the same way as the field ψ

$$\begin{aligned} D_\mu \psi(x) &\to \left[\partial_\mu + ie(A_\mu - \frac{1}{e}\partial_\mu \alpha)\right] e^{i\alpha(x)}\psi(x) \\ &= e^{i\alpha(x)}(\partial_\mu + ieA_\mu)\psi(x) = e^{i\alpha(x)} D_\mu \psi(x) \end{aligned} \tag{15.8}$$

Consider a locally invariant Lagrangian that depends on A_μ and its derivatives, re-define 15.5

$$U(x+\epsilon n, x) = \exp\left[-ie\epsilon n^\mu A_\mu(x + \frac{\epsilon}{2} n) + \mathcal{O}(\epsilon^3)\right] \tag{15.9}$$

then define $\mathbf{U}(x)$

$$\begin{aligned} \mathbf{U}(x) \equiv\, & U(x, x+\epsilon\hat{2})U(x+\epsilon\hat{2}, x+\epsilon\hat{1}+\epsilon\hat{2}) \\ & \times U(x+\epsilon\hat{1}+\epsilon\hat{2}, x+\epsilon\hat{1})U(x+\epsilon\hat{1}, x) \end{aligned} \tag{15.10}$$

inserting it into 15.9:

$$\begin{aligned} \mathbf{U}(x) = \exp\Big\{-ie\epsilon & \Big[-A_2(x+\frac{\epsilon}{2}\hat{2}) - A_1(x+\frac{\epsilon}{2}\hat{1}+\epsilon\hat{2}) \\ & +A_2(x+\epsilon\hat{1}+\frac{\epsilon}{2}\hat{2}) + A_1(x+\frac{\epsilon}{2}\hat{1})\Big] + \mathcal{O}(\epsilon^3)\Big\} \end{aligned} \tag{15.11}$$

Expand the exponent until second order

$$\mathbf{U}(x) = 1 - ie\epsilon^2\left[\partial_1 A_2(x) - \partial_2 A_1(x)\right] + \mathcal{O}(\epsilon^3) \tag{15.12}$$

Therefore $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is locally invariant, namely electromagnetic field tensor. In summary, from U(1) symmetry, we derive local invariance directly. Up to operators of dimension 4, there are only four possible terms inside Lagrangian invariant to global phase transformations.

$$\mathcal{L}_4 = \bar{\psi}(i\not{D})\psi - \frac{1}{4}(F_{\mu\nu})^2 - c\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} - m\bar{\psi}\psi \tag{15.17}$$

and we get Maxwell equations by Euler-Lagrange eq.

4.2 Yang-Mills Lagrangians

From single fermion field, to a doublet of Dirac fields

$$\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \tag{15.19}$$

which transform into one another under abstract three-dimensional rotations as a two-component spinor. (SU(2) symmetry)

$$\psi(x) \to \exp\left(i\alpha^i(x)\frac{\sigma^i}{2}\right)\psi(x) = V(x)\psi(x) \tag{15.20}$$

Similarly, define $U(y,x)$ as a 2×2 matrix.

$$U(y,x) \to V(y)U(y,x)V^\dagger(x) = \exp\left(i\alpha^i(y)\frac{\sigma^i}{2}\right)U(y,x)\exp\left(-i\alpha^i(x)\frac{\sigma^i}{2}\right) \tag{15.22}$$

again, set $U(y,y) = 1$, expand in therms of the Hermitian generators of SU(2) (Pauli matrices) with a constant g.

$$U(x+\epsilon n, x) = 1 + ig\epsilon n^\mu A^i_\mu \frac{\sigma^i}{2} + \mathcal{O}(\epsilon^2) \tag{15.23}$$

the covariant derivative associated with local SU(2) symmetry:

$$D_\mu = \partial_\mu - igA^i_\mu\frac{\sigma^i}{2} \tag{15.24}$$

What is non-abelian gauge? It's said that, there are not commutative property of multiplication for generators as matrices. Look for the connection A^i_μ liked 15.7 (see 15.27) and insert 15.23 into 15.22.

$$1 + ig\epsilon n^\mu A^i_\mu\frac{\sigma^i}{2} \to V(x+\epsilon n)\left(1 + ig\epsilon n^\mu A^i_\mu\frac{\sigma^i}{2}\right)V^\dagger(x) \tag{15.25}$$

$$\begin{aligned} V(x+\epsilon n)V^\dagger(x) &= \left[\left(1 + \epsilon n^\mu\frac{\partial}{\partial x^\mu} + \mathcal{O}(\epsilon^2)\right)V(x)\right]V^\dagger(x) \\ &= 1 + \epsilon n^\mu\left(\frac{\partial}{\partial x^\mu}V(x)\right) + \mathcal{O}(\epsilon^2) \\ &= 1 + \epsilon n^\mu V(x)\left(-\frac{\partial}{\partial x^\mu}V^\dagger(x)\right) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{15.26}$$

$$A^i_\mu(x)\frac{\sigma^i}{2} \to V(x)\left(A^i_\mu(x)\frac{\sigma^i}{2} + \frac{i}{g}\partial_\mu\right)V^\dagger(x) \tag{15.27}$$

Then insert it into 15.24, and the transformation law of covariant derivatives implies that

$$D_\mu\psi \to \left(1 + i\alpha^i\frac{\sigma^i}{2}\right)D_\mu\psi \tag{15.30}$$

$$[D_\mu, D_\nu]\,\psi(x) \to V(x)\,[D_\mu, D_\nu]\,\psi(x) \tag{15.31}$$

where $[D_\mu, D_\nu]$ is not a differential operator, merely a multiplicative matrix acting on $\psi(x)$.

$$[D_\mu, D_\nu] = -igF^i_{\mu\nu}\frac{\sigma^i}{2} \tag{15.32}$$

$$F^i_{\mu\nu}\frac{\sigma^i}{2} = \partial_\mu A^i_\nu \frac{\sigma^i}{2} - \partial_\nu A^i_\mu \frac{\sigma^i}{2} - ig\left[A^i_\mu \frac{\sigma^i}{2}, A^j_\nu \frac{\sigma^j}{2}\right] \tag{15.33}$$

Pauli matrices are generators of SU(2) symmetry, satisfying commutation relations,

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\epsilon^{ijk}\frac{\sigma^k}{2} \tag{15.34}$$

Then we get field strength tensor

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^j_\mu A^k_\nu \tag{15.35}$$

Now, field strength is not gauge invariant, with three possible directions denoted by i. Construct a locally invariant Lagrangian by combining field strengths:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\left[(F^i_{\mu\nu}\frac{\sigma^i}{2})^2\right] = -\frac{1}{4}(F^i_{\mu\nu})^2 \tag{15.38}$$

Note, in contrast to the case of electrodynamics, this Lagrangian contains cubic and quartic terms in A^i_μ. Thus, this Lagrangian describes a nontrivial, interacting field theory, called *Yang-Mills theory*. This is the simplest example of a non-Abelian gauge theory. And its Lagrangian looks like that of QED.

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi - \frac{1}{4}(F^i_{\mu\nu})^2 - m\bar{\psi}\psi \tag{15.39}$$

According to notions of wiki,

$$\mathcal{L}_{YM} \equiv -2\mathrm{Tr}(F \wedge *F) = -\frac{1}{4}(F^i_{\mu\nu})^2$$

- Quantum ElectroDynamics, $G = U(1)$
- ElectroWeak interaction, $G = U(1) \otimes SU(2)$
- Quantum ChromoDynamics, $G = SU(3)$
- Standard Model, $G = U(1) \otimes SU(2) \otimes SU(3)$

Part II

Chiral Perturbation Theory - Primer

Ref: "A Primer for Chiral Perturbation Theory" of Stefan Scherer & Matthias R. Schindler.

5 QCD & chiral symmetry

5.1 SU(3) group

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \lambda_0 = \sqrt{\frac{2}{3}} \cdot \mathbf{1}$$

These Gell-Mann matrices satisfy $\lambda_a = \lambda_a^\dagger$, $\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$, $\text{Tr}(\lambda_a) = 0$, as well as:

$$\left[\frac{\lambda_a}{2}, \frac{\lambda_b}{2}\right] = i f_{abc} \frac{\lambda_c}{2}$$

where f_{abc} is the coefficient:

$$f_{abc} = \frac{1}{4i} \text{Tr}([\lambda_a, \lambda_b]\, \lambda_c)$$

abc	123	147	156	246	257	345	367	458	678
f_{abc}	1	1/2	−1/2	1/2	1/2	1/2	−1/2	$\sqrt{3}/2$	$\sqrt{3}/2$

Like Dirac matrices γ^μ, eight λ_a are generators of SU(3) group, with unitary transformation $U^\dagger = U^{-1}$, where $\det(U) = 1$. We could derive $U(\Theta) = \exp\left(-i\Theta^a \lambda_a/2\right)$, as well as $\det\left[\exp(C)\right] = \exp\left[\text{Tr}(C)\right]$. (that's why generators are traceless)

The anticommutation relations of the Gell-Mann matrices read

$$\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} \cdot \mathbf{1} + 2 d_{abc} \lambda_c \tag{1.11}$$

where d_{abc} is totally symmetric

$$d_{abc} = \frac{1}{4} \text{Tr}\left\{\lambda_a, \lambda_b\right\} \lambda_c \tag{1.12}$$

Define $h_{abc} \equiv d_{abc} + i f_{abc}$, which is invariant under cyclic permutations, $h_{abc} = h_{bca} = h_{cab}$

$$\begin{aligned} \text{Tr}(\lambda_a \lambda_b \lambda_c) &= 2 h_{abc} \\ \text{Tr}(\lambda_a \lambda_b \lambda_c \lambda_d) &= \frac{4}{3}\delta_{ab}\delta_{cd} + 2 h_{abe} h_{ecd} \\ \text{Tr}(\lambda_a \lambda_b \lambda_c \lambda_d \lambda_e) &= \frac{4}{3} h_{abc}\delta_{de} + \frac{4}{3}\delta_{ab} h_{cde} + 2 h_{abf} h_{fcg} h_{gde} \end{aligned}$$

5.2 Lagrangians

$$\boxed{\mathscr{L}_{\text{QED}} = \bar{\Psi}\left[i\gamma^{\mu}\left(\partial_{\mu} - ie\mathscr{A}_{\mu}\right) - m\right]\Psi - \frac{1}{4}\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}}$$

the electromagnetic field-strength tensor is denoted by

$$\mathscr{F}_{\mu\nu} = \partial_{\mu}\mathscr{A}_{\nu} - \partial_{\nu}\mathscr{A}_{\mu}$$

Define the covariant derivative of Ψ

$$D_{\mu}\Psi \equiv \left(\partial_{\mu} - ie\mathscr{A}_{\mu}\right)\Psi$$

such that under a so-called gauge transformation of the second kind

$$\Psi(x) \to \exp\left[i\Theta(x)\right]\Psi(x) \qquad \mathscr{A}_{\mu}(x) \to \mathscr{A}_{\mu}(x) + \partial_{\mu}\Theta(x)/e$$

it transforms in the same way as Ψ,

$$D_{\mu}\Psi(x) \to \exp\left[i\Theta(x)\right]D_{\mu}\Psi(x)$$

... the interaction of the electromagnetic field:

$$\mathscr{L}_{\text{int}} = -(-e)\bar{\Psi}\gamma^{\mu}\Psi\mathscr{A}_{\mu} = -J^{\mu}\mathscr{A}_{\mu}$$

In QCD, $\alpha = 1, \cdots, 4$ refers to the Dirac-spinor index; $f = 1, \cdots, 6$, the flavor index; $A = 1, 2, 3$, the color index. The "free" quark Lagrangian w/o interaction may be written as

$$\mathscr{L}_{\text{free quarks}} = \sum_{A=1}^{3}\sum_{f=1}^{6}\sum_{\alpha,\alpha'=1}^{4}\bar{q}_{\alpha,f,A}\left(\gamma^{\mu}_{\alpha\alpha'}i\partial_{\mu} - m_f\delta_{\alpha\alpha'}\right)q_{\alpha',f,A} \tag{1.20}$$

Introduce a color triplet for each quark flavor f

$$q_f = \begin{pmatrix} q_{f,1} & q_{f,2} & q_{f,3} \end{pmatrix}^T$$

all q_f are subject to the same local SU(3) transformation; λ^c_a are Gell-Mann matrices in color space

$$q_f(x) \to q'_f(x) = \exp\left(i\Theta^a(x)\lambda^c_a/2\right)q_f(x) \equiv U(x)q_f(x)$$

To keep Lagrangians invarianct under local transformations, introduce eight 4-vector gauge potentials $\mathscr{A}_{a\mu}$, transforming as

$$\mathscr{A}_{\mu} \equiv \mathscr{A}_{a\mu}\frac{\lambda^c_a}{2} \to \mathscr{A}'_{\mu} = U\mathscr{A}_{\mu}U^{\dagger} + \frac{i}{g_3}\partial_{\mu}UU^{\dagger} \tag{1.24}$$

$\partial_{\mu}q_f$ is replaced by the covariant derivative, with the strong coupling constant g_3.

$$D_{\mu}q_f \equiv \left(\partial_{\mu} + ig_3\mathscr{A}_{\mu}\right)q_f$$

Define non-Abelian tensor,

$$\mathscr{G}_{a\mu\nu} \equiv \partial_\mu \mathscr{A}_{a\nu} - \partial_\nu \mathscr{A}_{a\mu} - g_3 f_{abc} \mathscr{A}_{b\mu} \mathscr{A}_{c\nu} \tag{1.26}$$

which under SU(3) transforms as

$$\mathscr{G}_{\mu\nu} \equiv \mathscr{G}_{a\mu\nu} \lambda_a^c / 2 \to U \mathscr{G}_{\mu\nu} U^\dagger \tag{1.27}$$

The QCD Lagrangian obtained by applying the gauge principle to the free Lagrangian eq. 1.20

$$\boxed{\mathscr{L}_{\mathrm{QCD}} = \sum_f \bar{q}_f \left(i \not{D} - m_f \right) - \frac{1}{4} \mathscr{G}_{a\mu\nu} \mathscr{G}_a^{\mu\nu}} \tag{1.28}$$

with the purely gluonic part $-\mathrm{Tr}\left(\mathscr{G}_{\mu\nu}\mathscr{G}^{\mu\nu}\right)$. The strong-interaction Lagrangian could also involve a term of the type

$$\mathscr{L}_\theta = \frac{g_3^2 \bar{\theta}}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathscr{G}_a^{\mu\nu} \mathscr{G}_a^{\rho\sigma}$$

5.3 Global Symmetries of the QCD Lagrangian

5.3.1 Light and Heavy Quarks

uds light, cbt heavy. $m_u = (1.7 - 3.3)\mathrm{MeV}$, $m_d = (4.1 - 5.8)\mathrm{MeV}$, $m_s = (80 - 130)\mathrm{MeV}$, $m_c = 1.27^{+0.07}_{-0.09}\mathrm{GeV}$, $m_b = 4.19^{+0.18}_{-0.06}\mathrm{GeV}$, $m_t = (172.0 \pm 0.9 \pm 1.3)\mathrm{GeV}$. Note that the mass of baryons might not equal to the sum of corresponding components. For example, $m_p = 938\mathrm{MeV} \gg 2m_u + m_d$. Here is the Lagrangian, containing only the light-flavor quarks in the so-called chiral limit $m_u, m_d, m_s \to 0$

$$\mathscr{L}^0_{\mathrm{QCD}} = \sum_{l=u,d,s} \bar{q}_l i \not{D} q_l - \frac{1}{4} \mathscr{G}_{a\mu\nu} \mathscr{G}_a^{\mu\nu}$$

5.3.2 Left-Handed and Right-Handed Quark Fields

Introduce the projection operators

$$P_R = (\mathbf{1} + \gamma^5)/2 = P_R^\dagger \qquad P_L = (\mathbf{1} - \gamma^5) = P_L^\dagger$$

with orthogonality relations:

$$P_L + P_R = \mathbf{1} \qquad P_{L/R}^2 = P_{L/R} \qquad P_{L/R} P_{R/L} = 0$$

which could project the variable q from the Dirac field, to its chiral components q_R and q_L;

$$q_R = P_R q \qquad\qquad q_L = P_L q$$

Quark field is transformed into its parity conjugate under parity, $P : q(t, \vec{x}) \to \gamma^0 q(t, -\vec{x})$; thus

$$q_R(t, \vec{x}) = P_R q(t, \vec{x}) \to P_R \gamma^0 q(t, -\vec{x}) = \gamma^0 P_L q(t, -\vec{x}) = \gamma^0 q_L(t, -\vec{x}) \neq \pm q_R(t, -\vec{x})$$

Suppose that the spin in the rest frame is (anti)parallel to the direction of momentum

$$\vec{\sigma} \cdot \hat{p}\chi_{\pm} = \pm\chi_{\pm}$$

$\bar{\psi}\Gamma\psi$ is Lorentz invariant

$$\bar{q}\Gamma q = \begin{cases} \bar{q}_R\Gamma q_R + \bar{q}_L\Gamma q_L & \text{for } \Gamma \in \Gamma_1 \equiv \{\gamma^\mu, \gamma^\mu\gamma_5\} \\ \bar{q}_R\Gamma q_L + \bar{q}_L\Gamma q_R & \text{for } \Gamma \in \Gamma_2 \equiv \{\mathbf{1}, \gamma_5, \sigma^{\mu\nu}\} \end{cases}$$

$$\bar{q}_R = q_R^\dagger\gamma_0 = q^\dagger P_R^\dagger\gamma_0 = q^\dagger P_R\gamma_0 = q^\dagger\gamma_0 P_L = \bar{q}P_L$$

Similarly $\bar{q}_L = \bar{q}P_R$. Under the chiral limit, we have the Lagrangian:

$$\mathscr{L}^0_{\text{QCD}} = \sum_{l=u,d,s} \left(\bar{q}_{R,l} i\not{D} q_{R,l} + \bar{q}_{L,l} i\not{D} q_{L,l}\right) - \frac{1}{4}\mathscr{G}_{a\mu\nu}\mathscr{G}_a^{\mu\nu}$$

Due to the flavor independence of the covariant derivative, $\mathscr{L}^0_{\text{QCD}}$ is invariant under

$$\begin{pmatrix} u_{L/R} \\ d_{L/R} \\ s_{L/R} \end{pmatrix} \to U_{L/R} \begin{pmatrix} u_{L/R} \\ d_{L/R} \\ s_{L/R} \end{pmatrix} = \exp\left(-i\sum_{a=1}^{8}\Theta^a_{L/R}\lambda_a/2\right) e^{-i\Theta_{L/R}} \begin{pmatrix} u_{L/R} \\ d_{L/R} \\ s_{L/R} \end{pmatrix}$$

5.3.3 Noether theorem

... introduce 3×3 matrices L_i^{ad} of the adjoint representation,

$$\hat{l}_i = -i\hat{p}_j(-i\epsilon_{ijk})\hat{x}_k = -i\hat{p}_j(L_i^{\text{ad}})_{jk}\hat{x}_k$$

Both the matrices of the adjoint representation and the components of the angular momentum operator satisfy the angular momentum commutation relations,

$$\left[L_i^{\text{ad}}, L_j^{\text{ad}}\right] = i\epsilon_{ijk}L_k^{\text{ad}} \qquad \left[\hat{l}_i, \hat{l}_j\right] = i\epsilon_{ijk}\hat{l}_k$$

... we consider the pseudoscalar pion-nucleon Lagrangian

$$\mathscr{L} = \bar{\Psi}(i\not{\partial} - m_N) + \frac{1}{2}\left(\partial_\mu\vec{\Psi}\cdot\partial^\mu\vec{\Psi} - M_\pi^2\vec{\Psi}^2\right) - ig\bar{\Psi}\gamma^5\vec{\Psi}\cdot\vec{\sigma}\Psi$$

5.3.4 Global Symmetry Currents of the Light-Quark Sector

$$\delta\mathscr{L}^0_{\text{QCD}} = \bar{q}_R\left(\sum_{a=1}^{8}\partial_\mu\epsilon_{Ra}\frac{\lambda_a}{2} + \partial_\mu\epsilon_R\right)\gamma^\mu q_R + \bar{q}_L\left(\sum_{a=1}^{8}\partial_\mu\epsilon_{La}\frac{\lambda_a}{2} + \partial_\mu\epsilon_L\right)\gamma^\mu q_L$$

according to $J_a^\mu = \frac{\partial\delta\mathscr{L}}{\partial\partial_\mu\epsilon_a}$, as well as $\partial_\mu J_a^\mu = \frac{\partial\delta\mathscr{L}}{\partial\epsilon_a}$, with the transformations of the left-handed or right-handed quarks

$$
\begin{array}{llll}
L_a^\mu = \frac{\partial \delta \mathscr{L}_{\rm QCD}^0}{\partial \partial_\mu \epsilon_{La}} = \bar{q}_L \gamma^\mu \frac{\lambda_a}{2} q_L & \partial_\mu L_a^\mu = \frac{\partial \delta \mathscr{L}_{\rm QCD}^0}{\partial \epsilon_{La}} = 0 \\
R_a^\mu = \frac{\partial \delta \mathscr{L}_{\rm QCD}^0}{\partial \partial_\mu \epsilon_{Ra}} = \bar{q}_R \gamma^\mu \frac{\lambda_a}{2} q_R & \partial_\mu R_a^\mu = \frac{\partial \delta \mathscr{L}_{\rm QCD}^0}{\partial \epsilon_{Ra}} = 0 \\
L^\mu = \frac{\partial \delta \mathscr{L}_{\rm QCD}^0}{\partial \partial_\mu \epsilon_{L}} = \bar{q}_L \gamma^\mu q_L & \partial_\mu L_a^\mu = \frac{\partial \delta \mathscr{L}_{\rm QCD}^0}{\partial \epsilon_{L}} = 0 \\
R^\mu = \frac{\partial \delta \mathscr{L}_{\rm QCD}^0}{\partial \partial_\mu \epsilon_{R}} = \bar{q}_R \gamma^\mu q_R & \partial_\mu R_a^\mu = \frac{\partial \delta \mathscr{L}_{\rm QCD}^0}{\partial \epsilon_{R}} = 0
\end{array}
$$

Instead of these chiral currents, we always use their linear combinations,

$$
V_a^\mu = R_a^\mu + L_a^\mu = \bar{q}\gamma^\mu \frac{\lambda_a}{2} q \qquad A_a^\mu = R_a^\mu - L_a^\mu = \bar{q}\gamma^\mu \gamma_5 \frac{\lambda_a}{2} q
$$

transforming under parity as vector and axial-vector currents,

$$
P : V_a^\mu(t, \vec{x}) \to V_{a\mu}(t, -\vec{x}) \qquad P : A_a^\mu(t, \vec{x}) \to -A_{a\mu}(t, -\vec{x})
$$

Similarly, $V^\mu = R^\mu + L^\mu = \bar{q}\gamma^\mu q$, $\partial_\mu V^\mu = 0$, $A^\mu = R^\mu - L^\mu = \bar{q}\gamma^\mu \gamma_5 q$,

$$
\partial_\mu A^\mu = \frac{3g_3^2}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathscr{G}_a^{\mu\nu} \mathscr{G}_a^{\rho\sigma}
$$

In the large N_c (number of color) limit, the strong coupling constant behaves as $g_3^2 \sim N_c^{-1}$.

5.3.5 Chiral Symmetry Breaking by the Quark Masses

For any 3×3 matrices, $M = \sum_{a=0}^{8} M_a \lambda_a$, $M_a = \frac{1}{2}\mathrm{Tr}(\lambda_a M)$. Consider the quark-mass matrix of the three light quarks and project it onto the nine λ matrices

$$
\mathscr{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}
$$

... applying Eq. 1.38 we see that the quark-mass term mixes left- and right-handed fields,

$$
\mathscr{L}_{\mathscr{M}} = -\bar{q}\mathscr{M} q = -(\bar{q}_R \mathscr{M} q_L + \bar{q}_L \mathscr{M} q_R)
$$

with related currents

$$
\partial_\mu L_a^\mu = \frac{\partial \delta \mathscr{L}_{\mathscr{M}}}{\partial \epsilon_{La}} = -i \left(\bar{q}_L \frac{\lambda_a}{2} \mathscr{M} q_R - \bar{q}_R \mathscr{M} \frac{\lambda_a}{2} q_L \right)
$$

$$
\partial_\mu R_a^\mu = \frac{\partial \delta \mathscr{L}_{\mathscr{M}}}{\partial \epsilon_{Ra}} = -i \left(\bar{q}_R \frac{\lambda_a}{2} \mathscr{M} q_L - \bar{q}_L \mathscr{M} \frac{\lambda_a}{2} q_R \right)
$$

$$
\partial_\mu L^\mu = \frac{\partial \delta \mathscr{L}_{\mathscr{M}}}{\partial \epsilon_L} = -i \left(\bar{q}_L \mathscr{M} q_R - \bar{q}_R \mathscr{M} q_L \right)
$$

$$
\partial_\mu R^\mu = \frac{\partial \delta \mathscr{L}_{\mathscr{M}}}{\partial \epsilon_R} = -i \left(\bar{q}_R \mathscr{M} q_L - \bar{q}_L \mathscr{M} q_R \right)
$$

5.4 Green Functions and Ward Identities

Ward identites of QED

$$\Gamma^\mu(p,p) = -\frac{\partial}{\partial p_\mu}\Sigma(p)$$

which relates the electromagnetic vertex of an electron at zero momentum transfer, $\gamma^\mu + \Gamma^\mu(p,p)$, to the electron self-energy, $\Sigma(p)$.

5.4.1 Ward Identities Resulting from U(1) Invariance

Focus on a scalar field theory w/ a global SO(2) or U(1) invariance. Consider a Lagrangian

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial_\mu\Phi_1\partial^\mu\Phi_1 + \partial_\mu\Phi_2\partial^\mu\Phi_2) - \frac{m^2}{2}(\Phi_1^2+\Phi_2^2) - \frac{\lambda}{4}(\Phi_1^2+\Phi_2^2)^2 \\ &= \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2\end{aligned} \tag{1.114}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2}}\left[\Phi_1(x) + i\Phi_2(x)\right] \qquad \Phi^\dagger(x) = \frac{1}{\sqrt{2}}\left[\Phi_1(x) - i\Phi_2(x)\right]$$

if $m^2 > 0$, $\lambda > 0$, so there is no spontaneous symmetry breaking. eq. 1.114 keeps invariant under global transformation

$$\Phi_1' = \Phi_1 - \epsilon\Phi_2 \qquad \Phi_2' = \Phi_2 + \epsilon\Phi_1 \tag{1.115}$$

equivalent to

$$\Phi' = (1+i\epsilon)\Phi \qquad \Phi'^\dagger(1-i\epsilon)\Phi^\dagger \tag{1.116}$$

with an infinitesimal real parameter ϵ. Apply the method of Gell-Mann and Lévy, we obtain for a *local* parameter $\epsilon(x)$;

$$\delta\mathscr{L} = \partial_\mu\epsilon(x)\left(i\partial^\mu\Phi^\dagger\Phi - i\Phi^\dagger\partial^\mu\Phi\right) \tag{1.117}$$

Since $J_a^\mu = \partial\delta\mathscr{L}/\partial\partial_\mu\epsilon_a$, and $\partial_\mu J_a^\mu = \partial\delta\mathscr{L}/\partial\epsilon_a$, we could derive

$$J^\mu = \frac{\partial\delta\mathscr{L}}{\partial\partial_\mu\epsilon} = i\partial^\mu\Phi^\dagger\Phi - i\Phi^\dagger\partial^\mu\Phi \tag{1.118}$$

$$\partial_\mu J^\mu = \frac{\partial\delta\mathscr{L}}{\partial\epsilon} = 0 \tag{1.119}$$

Consider general coordinates and general momenta

$$\Pi_i = \frac{\partial\mathscr{L}}{\partial\partial_0\Phi_i} \qquad \Pi = \frac{\partial\mathscr{L}}{\partial\partial_0\Phi} = \dot{\Phi}^\dagger \qquad \Pi^\dagger = \frac{\partial\mathscr{L}}{\partial\partial_0\Phi^\dagger} = \dot{\Phi} \tag{1.120}$$

with equal-time commutation relations

$$[\Phi_i(t,\vec{x}), \Pi_j(t,\vec{y})] = i\delta_{ij}\delta^3(\vec{x}-\vec{y}) \tag{1.121}$$

A conserved current is time-independent

$$Q = \int d^3x J^0(t,\vec{x}) \tag{1.124}$$

Applying eq. 1.122, we have

$$\begin{aligned}
\left[J^0(t,\vec{x}),\Phi(t,\vec{y})\right] &= \delta^3(\vec{x}-\vec{y})\Phi(t,\vec{x}) \\
\left[J^0(t,\vec{x}),\Pi(t,\vec{y})\right] &= -\delta^3(\vec{x}-\vec{y})\Pi(t,\vec{x}) \\
\left[J^0(t,\vec{x}),\Phi^\dagger(t,\vec{y})\right] &= -\delta^3(\vec{x}-\vec{y})\Phi^\dagger(t,\vec{x}) \\
\left[J^0(t,\vec{x}),\Pi^\dagger(t,\vec{y})\right] &= \delta^3(\vec{x}-\vec{y})\Pi^\dagger(t,\vec{x})
\end{aligned} \tag{1.125}$$

or the space integrals

$$\begin{aligned}
[Q,\Phi(x)] &= \Phi(x) \\
[Q,\Pi(x)] &= -\Pi(x) \\
\left[Q,\Phi^\dagger(x)\right] &= -\Phi^\dagger(x) \\
\left[Q,\Pi^\dagger(x)\right] &= \Pi^\dagger(x)
\end{aligned} \tag{1.126}$$

Suppose $|\alpha\rangle$ is the eigenstate of Q, with eigenvalue q_α

$$Q(\Phi(x)|\alpha\rangle) = ([Q,\Phi(x)]+\Phi(x)Q)|\alpha\rangle = (1+q_\alpha)(\Phi(x)|\alpha\rangle)$$

In summary, $\Phi(x),\Pi^\dagger(x)$ increase the Noether charge of a system by one unit; $\Phi^\dagger(x),\Pi(x)$ decrease the Noether charge of a system by one unit.

Back to 1.114, discuss the consequences of the U(1) symmetry. Consider such a Green function,

$$G^\mu(x,y,z) = \langle 0|T\left[\Phi(x)J^\mu(y)\Phi^\dagger(z)\right]|0\rangle \tag{1.127}$$

which describes the transition amplitude for the creation of a quantum of Noether charge +1 at x, propagation to y, interaction at y via the current operator, propagation to z with annihilation at z. $|0\rangle$ refers to the ground state of the quantum field theory described by the Lagrangian of Eq. 1.114 (better written as $|\Omega\rangle$), and should not be confused with the ground state of a free theory.

$$\partial^y_\mu G^\mu(x,y,z) = \left[\delta^4(y-x)-\delta^4(y-z)\right]\langle 0|T\left[\Phi(x)\Phi^\dagger(z)\right]|0\rangle \tag{1.129}$$

eq. 1.129 is the analogue of the Ward identity of QED. (Current conserved, $[Q,J^\mu(x)]=0$.)

The Green function remains invariant under the U(1) transformation.

$$\begin{aligned}
G^\mu(x,y,z)\to G'^\mu(x,y,z) &= \langle 0|T\left[(1+i\epsilon)\Phi(x)J'^\mu(y)(1-i\epsilon)\Phi^\dagger(z)\right]|0\rangle \\
&= \langle 0|T\left[\Phi(x)J^\mu(y)\Phi^\dagger(z)\right]|0\rangle \\
&= G^\mu(x,y,z)
\end{aligned} \tag{1.128}$$

The underlying symmetry not only determines the transformation behavior of Green functions under the group, but also relates n-point Green functions containing a symmetry current to $(n-1)$-point Green functions. In principle, calculations similar to those leading to 1.128 and 1.129, can be performed for any Green function of the theory.

$$\begin{aligned}
\langle 0|T\left[\Phi(x)J^\mu(y)\Phi^\dagger(z)\right]|0\rangle &= \Phi(x)J^\mu(y)\Phi^\dagger(z)\Theta(x_0-y_0)\Theta(y_0-z_0) \\
&\quad + \Phi(x)\Phi^\dagger(z)J^\mu(y)\Theta(x_0-z_0)\Theta(z_0-y_0)+\cdots
\end{aligned}$$

All in all there exists 3!=6 distinct orderings. And $\partial^y_\mu\Theta(x_0-y_0) = -g_{\mu 0}\delta(x_0-y_0)$.

We will now show that the symmetry constraints imposed by the Ward identities can be compactly summarized in terms of an invariance property of a generating functional. (See Appendix B) The generating functional depends on a set of functions denoted by j, j^*, j_μ (external sources). They couple to the fields $\Phi^\dagger, \Phi, J^\mu$, respectively. The generating functional is defined as

$$\begin{aligned} W\left[j, j^*, j_\mu\right] &= \exp(iZ\left[j, j^*, j_\mu\right]) \\ &= \left\langle \Omega \middle| T\left(\exp\left\{ i\int d^4x \left[j(x)\Phi^\dagger(x) + j^*(x)\Phi(x) + j_\mu(x)J^\mu(x)\right]\right\}\right) \middle| \Omega \right\rangle \end{aligned} \tag{1.130}$$

$\Phi^\dagger, \Phi, J^\mu$ refer to the field operators and the Noether current in the Heisenberg picture, satisfying the Heisenberg equations of motion:

$$\begin{aligned} \partial_0\Phi(x) &= i\left[H, \Phi(x)\right] \\ \partial_0\Pi^\dagger(x) &= i\left[H, \Pi^\dagger(x)\right] \\ \partial_0\Phi^\dagger(x) &= i\left[H, \Phi^\dagger(x)\right] \\ \partial_0\Pi(x) &= i\left[H, \Pi(x)\right] \end{aligned} \tag{1.131}$$

where

$$H = \int d^3x \mathscr{H} \tag{1.132}$$

$$\mathscr{H} = \Pi^\dagger\Pi + \vec{\nabla}\Phi^\dagger \cdot \vec{\nabla}\Phi + m^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 \tag{1.133}$$

Via the equations of motion, generating functional depends on the dynamics of the system (eq. 1.114 & 1.132). Green function is partial derivative of generating functional

$$G^\mu(x, y, z) = (-i)^3 \left. \frac{\delta^3 W\left[j, j^*, j_\mu\right]}{\delta j^*(x)\delta j_\mu(y)\delta j(z)} \right|_{j=0, j^*=0, j_\mu=0} \tag{1.134}$$

Also, the generating functional may be written as the vacuum-to-vacuum transition amplitude in the presence of external fields,

$$W\left[j, j^*, j_\mu\right] = \langle 0, \text{out}|0, \text{in}\rangle_{j, j^*, j_\mu} \tag{1.135}$$

Consider the path integral below

$$W\left[j, j^*, j_\mu\right] = \int \left[d\Phi_1\right]\left[d\Phi_2\right] e^{iS\left[\Phi, \Phi^*, j, j^*, j_\mu\right]} \tag{1.136}$$

where

$$S\left[\Phi, \Phi^*, j, j^*, j_\mu\right] = S\left[\Phi, \Phi^*\right] + \int d^4x \left[\Phi(x)j^*(x) + \Phi^*(x)j(x) + J^\mu(x)j_\mu(x)\right] \tag{1.137}$$

denotes the action corresponding to the Lagrangian of eq. 1.114 in combination with a coupling to the external sources. In the path integral formulation we deal with functional integrals instead of linear operators. in the following we will write Φ^* instead of $\Phi^\dagger$... ...

5.4.2 Chiral Green Functions

In addition to the vector and axial-vector currents of eqs. 1.92, 1.93, 1.96, we want to investigate scalar and pseudo-scalar densities, ($a = 0, \cdots, 8$ shown below)

$$S_a(x) = \bar{q}(x)\lambda_a q(x) \qquad P_a(x) = i\bar{q}(x)\gamma_5\lambda_a q(x) \tag{1.143}$$

... ... Whenever it is more convenient, we will also use eqs below instead of S_0 or P_0

$$S(x) = \bar{q}(x)q(x) \qquad P(x) = i\bar{q}(x)\gamma_5 q(x) \tag{1.144}$$

... the following Green functions of the "vacuum" sector,

$$\begin{aligned} &\langle 0|T\left[A_a^\mu(x)P_b(y)\right]|0\rangle \\ &\langle 0|T\left[P_a(x)J^\mu(y)P_b(z)\right]|0\rangle \\ &\langle 0|T\left[P_a(w)P_b(x)P_c(y)P_d(z)\right]|0\rangle \end{aligned}$$

are related to 1. pion decay; 2. the pion electromagnetic form factor (J^μ is the electromagnetic current); and 3. pion-pion scattering, respectively. One may also consider similar time-ordered products evaluated between a single nucleon in the initial and final states in addition to the vacuum Green functions. The allows one to discuss properties of the nucleon as well as dynamical processes involving a single nucleon, such as

- $\langle N|J^\mu(x)|N\rangle \leftrightarrow$ nucleon electromagnetic form factors,
- $\langle N|A_a^\mu(x)|N\rangle \leftrightarrow$ axial form factor + induced pseudoscalar form factor,
- $\langle N|T\left[J^\mu(x)J^\nu(y)\right]|N\rangle \leftrightarrow$ Compton scattering,
- $\langle N|T\left[J^\mu(x)P_a(y)\right]|N\rangle \leftrightarrow$ pion photo- and electroproduction,

... consider a simple example the two-point Green function involving an axial-vector current and a pseudo-scalar density

$$\begin{aligned} G^\mu_{APab}(x,y) &= \langle 0\left|T\left[A_a^\mu(x)P_b(y)\right]\right|0\rangle \\ &= \Theta(x_0-y_0)\langle 0\left|A_a^\mu(x)P_b(y)\right|0\rangle + \Theta(y_0-x_0)\langle 0\left|P_b(y)A_a^\mu(x)\right|0\rangle \end{aligned} \tag{1.145}$$

and evaluate the divergence

$$\begin{aligned} \partial_\mu^x G^\mu_{APab}(x,y) &= \partial_\mu^x\left[\Theta(x_0-y_0)\langle 0\left|A_a^\mu(x)P_b(y)\right|0\rangle + \Theta(y_0-x_0)\langle 0\left|P_b(y)A_a^\mu(x)\right|0\rangle\right] \\ &= \delta(x_0-y_0)\left\langle 0\left|A_a^0(x)P_b(y)\right|0\right\rangle - \delta(x_0-y_0)\left\langle 0\left|P_b(y)A_a^0(x)\right|0\right\rangle \\ &\quad + \Theta(x_0-y_0)\left\langle 0\left|\partial_\mu^x A_a^\mu(x)P_b(y)\right|0\right\rangle + \Theta(y_0-x_0)\left\langle 0\left|P_b(y)\partial_\mu^x A_a^\mu(x)\right|0\right\rangle \\ &= \delta(x_0-y_0)\left\langle 0\left|\left[A_a^0(x),P_b(y)\right]\right|0\right\rangle + \left\langle 0\left|T\left[\partial_\mu^x A_a^\mu(x)P_b(y)\right]\right|0\right\rangle \end{aligned}$$

where we made use of $\partial_\mu^x\Theta(x_0-y_0) = \delta(x_0-y_0)g_{0\mu} = -\partial_\mu^x\Theta(y_0-x_0)$.

5.4.3 QCD in the Presence of External Fields and the Generating Functional

Introduce the couplings of the 9 vector currents and the 8 axial-vector currents into the Lagrangian of QCD, and the scalar and pseudoscalar quark densities to external c-number fields,

$$\mathscr{L} = \mathscr{L}^0_{\text{QCD}} + \mathscr{L}_{\text{ext}} \tag{1.150}$$

where

$$\begin{aligned}\mathscr{L}_{\text{ext}} &= \sum_{a=1}^{8} v^\mu_a \bar{q}\gamma_\mu \frac{\lambda_a}{2} q + v^\mu_{(s)} \frac{1}{3}\bar{q}\gamma_\mu q + \sum_{a=1}^{8} a^\mu_a \bar{q}\gamma_\mu\gamma_5 \frac{\lambda_a}{2} q - \sum_{a=0}^{8} s_a \bar{q}\lambda_a q + \sum_{a=0}^{8} p_a i\bar{q}\gamma_5\lambda_a q \\ &= \bar{q}\gamma_\mu \left(v^\mu + \frac{1}{3} v^\mu_{(s)} + \gamma_5 a^\mu \right) q - \bar{q}(s - i\gamma_5 p) q\end{aligned} \tag{1.151}$$

with 35 real functions of x: v^μ_a, $v^\mu_{(s)}$, a^μ_a, s_a, p_a, denoted by $[v, a, s, p]$; v,a,s,p refer to vector, axial-vector, scalar, pseudo-scalar current. The ordinary three-flavor QCD Lagrangian is recovered by setting $v^\mu = v^\mu_{(s)} = a^\mu = p = 0$, and $s = \text{diag}(m_u, m_d, m_s)$. The Green functions of the vacuum sector may be combined in the generating functional (where sub 0 means the chiral limit.)

$$\boxed{\exp(iZ[v, a, s, p]) = \left\langle 0 \left| T \exp\left[i \int d^4x \mathscr{L}_{\text{ext}}(x) \right] \right| 0 \right\rangle_0 = \langle 0, \text{out} | 0, \text{in} \rangle_{v,a,s,p}} \tag{1.152}$$

The quark fields are operators in the Heisenberg picture and have to satisfy the equations of motion and the canonical anticommutation relations. The generating functional is related to the vac-to-vac transition amplitude in the presence of external fields,

$$\exp(iZ[v, a, s, p]) = \langle 0, \text{out} | 0, \text{in} \rangle_{v,a,s,p} \tag{1.153}$$

For example, to calculate $\langle 0|\bar{u}u|0\rangle_0$, start from $\bar{u}u$

$$\bar{u}u = \frac{1}{2}\sqrt{\frac{2}{3}}\bar{q}\lambda_0 q + \frac{1}{2}\bar{q}\lambda_3 q + \frac{1}{2}\sqrt{\frac{1}{3}}\bar{q}\lambda_8 q$$

then

$$\langle 0|\bar{u}(x)u(x)|0\rangle_0 = \frac{i}{2}\left[\sqrt{\frac{2}{3}} \frac{\delta}{\delta s_0(x)} + \frac{\delta}{\delta s_3(x)} + \sqrt{\frac{1}{3}} \frac{\delta}{\delta s_8(x)} \right] \exp(iZ[v, a, s, p])|_{v=a=s=p=0}$$

The actual value of the generating functional for a given configuration of external field v, a, s, p reflects the dynamics generated by the QCD Lagrangian. The (infinite) set of *all* chiral Ward identities resides in an invariance of the generating functional under a *local* transformation of the external fields.

Γ	$\mathbf{1}$	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$\gamma_0\Gamma\gamma_0$	$\mathbf{1}$	γ_μ	$\sigma_{\mu\nu}$	$-\gamma_5$	$-\gamma_\mu\gamma_5$

Parity transformation of Γmatrices

$$q_f(t,\vec{x}) \xrightarrow{P} \gamma_0 q_f(t,-\vec{x}) \qquad \mathscr{L}(t,\vec{x}) \xrightarrow{P} \mathscr{L}(t,-\vec{x})$$

$$v^\mu \xrightarrow{P} v_\mu \qquad v^\mu_{(s)} \xrightarrow{P} v^{(s)}_\mu \qquad a^\mu \xrightarrow{P} -a_\mu \qquad s \xrightarrow{P} s \qquad p \xrightarrow{P} -p$$

Charge conjugate transformation of Γmatrices

Γ	$\mathbf{1}$	γ^μ	$\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$
$-C\Gamma^T C$	$\mathbf{1}$	$-\gamma^\mu$	$-\sigma^{\mu\nu}$	γ_5	$\gamma^\mu\gamma_5$

$$q_{\alpha,f} \xrightarrow{C} C_{\alpha\beta}\bar{q}_{\beta,f} \qquad \bar{q}_{\alpha,f} \xrightarrow{C} -q_{\beta,f}C^{-1}_{\beta\alpha}$$

$$C = i\gamma^2\gamma^0 = -C^{-1} = -C^\dagger = -C^T = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

F denotes flavor space matrix

$$\bar{q}\Gamma F q \xrightarrow{C} -\bar{q}C\Gamma^T C F^T q$$

Also for vasp

$$v_\mu \xrightarrow{C} -v_\mu^T \qquad v^{(s)}_\mu \xrightarrow{C} -v^{(s)T}_\mu \qquad a_\mu \xrightarrow{C} a_\mu^T \qquad s \xrightarrow{C} s^T \qquad p \xrightarrow{C} p^T$$

Define

$$r_\mu = v_\mu + a_\mu \qquad l_\mu = v_\mu - a_\mu \tag{1.160}$$

Eq. 1.151 becomes

$$\begin{aligned} \mathscr{L} =& \mathscr{L}^0_{\text{QCD}} + \bar{q}_L\gamma^\mu\left(l_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_L + \bar{q}_R\gamma^\mu\left(r_\mu + \frac{1}{3}v^{(s)}_\mu\right)q_R \\ & - \bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R \end{aligned} \tag{1.161}$$

which keeps invariant under local transformations (from now on V_R and V_L will denote local transformations, whereas R and L will be used for global transformations; V_R and V_L are independent space-time-dependent SU(3) matrices......)

$$q_{L/R} \to \exp\left(-i\Theta(x)/3\right) V_{L/R}(x) q_{L/R}$$

and for other parameters

$$\begin{aligned} r_\mu &\to V_R r_\mu V_R^\dagger + iV_R\partial_\mu V_R^\dagger \\ l_\mu &\to V_L l_\mu V_L^\dagger + iV_L\partial_\mu V_L^\dagger \\ v^{(s)}_\mu &\to v^{(s)}_\mu - \partial_\mu\Theta \\ s+ip &\to V_R(s+ip)V_L^\dagger \\ s-ip &\to V_L(s-ip)V_R^\dagger \end{aligned} \tag{1.163}$$

Coupling with external fields, we select the charge of quark as $Q = \mathrm{diag}(2/3, -1/3, -1/3)$

$$r_\mu = l_\mu = -e\mathscr{A}_\mu Q$$

$$\begin{aligned}\mathscr{L}_{\text{ext}} &= -e\mathscr{A}_\mu(\bar{q}_L Q\gamma^\mu q_L + \bar{q}_R Q\gamma^\mu q_R) = -e\mathscr{A}_\mu \bar{q} Q\gamma^\mu q \\ &= -e\mathscr{A}_\mu\left(\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d - \frac{1}{3}\bar{s}\gamma^\mu s\right) = -e\mathscr{A}_\mu J^\mu\end{aligned}$$

if only consider two-flavor QCD:

$$r_\mu = l_\mu = -e\mathscr{A}_\mu \frac{\sigma^3}{2} \qquad v_\mu^{(s)} = -\frac{e}{2}\mathscr{A}_\mu$$

6 Spontaneous Symmetry Breaking & the Goldstone Theorem

6.1 Degenerate Ground States

Consider the Lagrangian of a real scalar field $\Phi(x)$

$$\mathscr{L}(\Phi, \partial_\mu\Phi) = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - \frac{\lambda}{4}\Phi^4$$

By Legrendre transformation, the corresponding classical energy density reads,

$$\mathscr{H} = \Pi\dot{\Phi} - \mathscr{L} = \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}(\vec{\nabla}\Phi)^2 + \underbrace{\frac{m^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4}_{\equiv\mathscr{V}(\Phi)}$$

6.2 Continuous, Non-Abelian Symmetry

...

6.3 Goldstone Theorem

6.4 Explicit Symmetry Breaking

7 Chiral Perturbation Theory for Mesons

7.1 Effective Field Theory

7.2 Spontaneous Symmetry Breaking in QCD

7.3 Transformation Properties of the Goldstone Bosons

7.3.1 General Considerations

Let's consider a physical system described by a Lagrangian which is invariant under a compact Lie group G. We assume the ground state of the system to be invariant under only a subgroup H of G; giving rise to $n = n_G - n_H$ Goldstone bosons. Each of these Goldstone bosons will be described by an independent field ϕ_i which is a smooth real function on Minkowski space M^4. These fields are collected in an n-component vector $\Phi = (\phi_1, \cdots, \phi_n)$. Define the real vector space

$$M_1 \equiv \{\Phi : M^4 \to \mathbb{R}^n | \phi_i : M^4 \to \mathbb{R} \text{ smooth}\}$$

to seek a mapping φ which uniquely associates with each pair $(g, \Phi) \in G \times M_1$ an element $\varphi(g, \Phi) \in M_1$, with following properties

$$\varphi(e, \Phi) = \Phi \ (\forall \Phi \in M_1)$$

$$\varphi(g_1, \varphi(g_2, \Phi)) = \varphi(g_1 g_2, \Phi) \ (\forall g_1, g_2 \in G, \forall \Phi \in M_1)$$

Such a mapping defines an *operation* of the group G on M_1.

$$\begin{array}{ccc} \Phi & \xrightarrow{g} & \Phi' \\ \downarrow & & \uparrow \\ \tilde{g}H & \xrightarrow{g} & g\tilde{g}H \end{array}$$

This procedure uniquely determines the transformation behavior of the Goldstone bosons up to an appropriate choice of variables parameterizing the elements of the quotient G/H.

7.3.2 Application to QCD

The symmetry groups relevant to the application in QCD are ($N_C = 2$ for ud, $N_C = 3$ for uds)

$$G = \mathrm{SU}(N) \times \mathrm{SU}(N) = \{(L, R) | L \in \mathrm{SU}(N), R \in \mathrm{SU}(N)\}$$

$$H = \{(V, V) | V \in \mathrm{SU}(N)\} \cong \mathrm{SU}(N)$$

The transformation behavior of U is therefore given by

$$U = \tilde{R}\tilde{L}^\dagger \to U' = R(\tilde{R}\tilde{L}^\dagger)L^\dagger = RUL^\dagger$$

Re-define

$$M_1 \equiv \begin{cases} \{\Phi : M^4 \to \mathbb{R}^3 | \phi_i : M^4 \to \mathbb{R} \text{ smooth}\} \text{ for} N = 2 \\ \{\Phi : M^4 \to \mathbb{R}^8 | \phi_i : M^4 \to \mathbb{R} \text{ smooth}\} \text{ for} N = 3 \end{cases}$$

let $\tilde{\mathscr{H}}(N)$ denote the set of all Hermitian and traceless $N \times N$ matrices, which under addition of matrices defines a real vector space.

$$\tilde{\mathscr{H}}(N) \equiv \{A \in \mathrm{GL}(N, \mathbb{C}) | A^\dagger = A \wedge \mathrm{Tr}(A) = 0\}$$

Define $M_2 \equiv \{\phi : M^4 \to \tilde{\mathscr{H}}(N) | \phi \text{ smooth}\}$, if $N = 2$:

$$\phi = \sum_{i=1}^{3} \phi_i \sigma_i = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix} \equiv \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \tag{3.36}$$

where σ^i is Pauli matrix, and $\phi_i = \frac{1}{2}\mathrm{Tr}(\sigma_i \phi)$. If $N = 3$,

$$\phi = \sum_{i=1}^{8} \phi_i \lambda^i \equiv \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix} \tag{3.37}$$

Similarly, λ_a is Gell-Mann matrix, and $\phi_a = \frac{1}{2}\mathrm{Tr}(\lambda_a \phi)$. Therefore, $\pi^+ = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$

Finally, we define

$$M_3 \equiv \left\{ U : M^4 \to SU(N) | U = \exp\left(i\frac{\phi}{F_0}\right), \phi \in M_2 \right\}$$

F_0 is introduced to make the argument of the exponential function dimensionless. Since a bosonic field has the dimension of energy, F_0 also has the dimension of energy. Later on, F_0 will be identified with the "decay" constant of the Goldstone bosons in the chiral limit. (There is a subtlety here, because F_0 is traditionally reserved for the three-flavor chiral limit, whereas the two-flavor chiral limit and at fixed m_s is denoted by F.) ......

$$U = \mathbf{1} + i\frac{\phi}{F_0} - \frac{\phi^2}{2F_0^2} + \cdots$$

7.4 Effective Lagrangian and Power-Counting Scheme

Mainly for couplings with external fields under 1GeV

7.4.1 The Lowest-Order Effective Lagrangian

In the chiral limit, effective Lagrangian is invariant under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V$. It should contain exactly eight pseudoscalar degrees of freedom transforming as an octet under the subgroup $H = \mathrm{SU}(3)_V$. In terms of the SU(3) matrix, with an octet ϕ

$$U(x) = \exp\left(i\frac{\phi(x)}{F_0}\right) \tag{3.41}$$

the most general, chirally invariant, effective Lagrangian with the minimal number of derivatives reads

$$\mathscr{L}_{\text{eff}} = \frac{F_0^2}{4}\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) \qquad L/R = \mathbf{1} - i\epsilon^a_{L/R}\lambda_a/2$$

7.4.2 Symmetry Breaking by the Quark Masses

Consider the isospin-symmetric limit, $m_u = m_d = \hat{m}$, so that the $\pi^0\eta$ term vanishes and there is no $\pi^0\eta$ mixing. Then we obtain for the masses of the Goldstone bosons, to lowest order in the quark masses, with $B_0 = -\langle\bar{q}q\rangle/(3F_0^2)$

$$M_\pi^2 = 2B_0\hat{m} \qquad M_K^2 = B_0(\hat{m}+m_s) \qquad M_\eta^2 = \frac{2}{3}B_0(\hat{m}+2m_s)$$

7.4.3 Construction of the Effective Lagrangian

As in the case of gauge theories, we need external fields $l_a^\mu(x)$ and $r_a^\mu(x)$ (Eqs. 1.151, 1.160, 1.163, Table 3.3). For any object A (like U), transforming as $V_R A V_L^\dagger$, define the covariant derivative $D_\mu A$:

$$D_\mu A \equiv \partial_\mu A - ir_\mu A + iAl_\mu \tag{3.65}$$

Also, there are the field-strength tensors $f_{L\mu\nu}$ and $f_{R\mu\nu}$

$$f_{R\mu\nu} \equiv \partial_\mu r_v - \partial_v r_\mu - i\left[r_\mu, r_\nu\right] \qquad f_{L\mu\nu} \equiv \partial_\mu l_v - \partial_v l_\mu - i\left[l_\mu, l_\nu\right]$$

field strength tensors are traceless. Introduce linear combination $\chi \equiv 2B_0(s+ip)$, where

$$3F_0^2 B_0 = -\langle\bar{q}q\rangle_0$$

As for ChPT,

$$U = \mathcal{O}(q^0) \qquad D_\mu U = \mathcal{O}(q) \qquad l/r_\mu = \mathcal{O}(q) \qquad f_{L/R\mu\nu} = \mathcal{O}(q^2) \qquad \chi = \mathcal{O}(q^2)$$

with invariances below

$$\begin{aligned}
\mathcal{O}(q^0) &: \mathrm{Tr}(UU^\dagger) = 3 \\
\mathcal{O}(q) &: \mathrm{Tr}(D_\mu U U^\dagger) \overset{*}{=} -\mathrm{Tr}\left[U(D_\mu U)^\dagger\right] \overset{*}{=} 0 \\
\mathcal{O}(q^2) &: \mathrm{Tr}(D_\mu D_\nu U U^\dagger) \overset{**}{=} -\mathrm{Tr}\left[D_\nu U(D_\mu U)^\dagger\right] \overset{**}{=} \mathrm{Tr}\left[U(D_\nu D_\mu U)^\dagger\right] \\
&\quad \mathrm{Tr}(\chi U^\dagger) \\
&\quad \mathrm{Tr}(U\chi^\dagger) \\
&\quad \mathrm{Tr}(U f_{L\mu\nu} U^\dagger) = \mathrm{Tr}(f_{L\mu\nu}) = 0 \\
&\quad \mathrm{Tr}(f_{R\mu\nu}) = 0
\end{aligned} \tag{3.72}$$

We end up with the most general, locally invariant, effective Lagrangian at lowest chiral order

$$\mathscr{L}_2 = \frac{F_0^2}{4}\mathrm{Tr}\left[D_\mu U(D^\mu U)^\dagger\right] + \frac{F_0^2}{4}\mathrm{Tr}(\chi U^\dagger + U\chi^\dagger)$$

At $\mathcal{O}(q^2)$ it contains two-low energy constants: the SU(3) chiral limit of the Goldstone=boson decay constant F_0, and $B_0 = -\langle 0|\bar{q}q|0\rangle_0/(3F_0^2)$

8 Chiral Perturbation Theory for Baryons

Page 159, eq. 4.13, chiral connection Γ_μ:

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right]$$

Part III
omega to pi + gamma

Ref: Pedro D. Ruiz-Femenía et al JHEP07(2003)003

9 Symmetry

Operators	Parity trans$P$	Charge$C$	Hermitian conjugate
Φ	$-\Phi$	Φ^T	Φ
u	$u^\dagger$	u^T	$u^\dagger$
$D_\mu u$	$(D_\mu u)^\dagger$	$(D_\mu u)^T$	$(D_\mu u)^\dagger$
v_μ	v^μ	$-v_\mu^T$	v_μ
a_μ	$-a^\mu$	a_μ^T	a_μ
l_μ	r^μ	$-r_\mu^T$	l_μ
r_μ	l^μ	$-l_\mu^T$	r_μ
χ	$\chi^\dagger$	χ^T	$\chi^\dagger$
$F_L^{\mu\nu}$	$F_{R\mu\nu}$	$-F_R^{\mu\nu T}$	$F_L^{\mu\nu}$
$F_R^{\mu\nu}$	$F_{L\mu\nu}$	$-F_L^{\mu\nu T}$	$F_R^{\mu\nu}$
u_μ	$-u^\mu$	$u^{\mu T}$	u_μ
$\chi_\pm$	$\pm\chi_\pm$	$\chi_\pm^T$	$\pm\chi_\pm$
Γ_μ	Γ^μ	Γ_μ^T	$-\Gamma_\mu$
$f_{\mu\nu\pm}$	$\pm f_\pm^{\mu\nu}$	$\mp f_{\mu\nu\pm}^T$	$f_{\mu\nu\pm}$
$h_{\mu\nu}$	$-h^{\mu\nu}$	$h_{\mu\nu}^T$	$h_{\mu\nu}$
S	S	S^T	S
P	$-P$	P^T	P
$V_{\mu\nu}$	$V^{\mu\nu}$	$-V_{\mu\nu}^T$	$V_{\mu\nu}$
$A_{\mu\nu}$	$-A^{\mu\nu}$	$A_{\mu\nu}^T$	$A_{\mu\nu}$

10 Notions

Ref, "S. Scherer Introduction to Chiral Perturbation Theory" on page 49.
We introduce into the Lagrangian of QCD the couplings of the nine vector currents and the eight axial-vector currents as well as the scalar and pseudoscalar quark densities to external c-number fields $v^\mu(x)$, $v^\mu_{(s)}$, $a^\mu(x)$, $s(x)$, $p(x)$.

$$\mathscr{L}_{\text{ext}} = \bar{q}\gamma_\mu \left(v^\mu + \frac{1}{3} v^\mu_{(s)} + \gamma_5 a^\mu \right) q - \bar{q}(s - i\gamma_5 p)q$$

The external fields are color-neutral, Hermitian 3×3 matrices, where the matrix character, with respect to the (suppressed) flavor indices uds of the quarks fields, is eq. 2.96

$$v^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} v_a^\mu \qquad a^\mu = \sum_{a=1}^{8} \frac{\lambda_a}{2} a_a^\mu \qquad s = \sum_{a=0}^{8} \lambda_a s_a \qquad p = \sum_{a=0}^{8} \lambda_a p_a$$

The ordinary three flavor QCD Lagrangian is recovered by setting $v^\mu = v^\mu_{(s)} = a^\mu = p = 0$ and $s = \mathrm{diag}(m_u, m_d, m_s)$ Note the couplings of quarks with external fields, we choose the charge as $Q = \mathrm{diag}(2/3, -1/3, -1/3)$. As for three-flavor quarks, $r_\mu = l_\mu = -e\mathscr{A}_\mu Q$

$$r_\mu = v_\mu + a_\mu \qquad\qquad l_\mu = v_\mu - a_\mu$$

10.1 Field strength tensor $F^{\mu\nu}_{L/R}$

$$F^{\mu\nu}_R = \partial_\mu r_\nu - \partial_\nu r_\mu - i\,[r_\mu, r_\nu] \qquad\qquad F^{\mu\nu}_L = \partial_\mu l_\nu - \partial_\nu l_\mu - i\,[l_\mu, l_\nu]$$

with external fields $r_\mu = v_\mu + a_\mu$, $l_\mu = v_\mu - a_\mu$;

$$\begin{aligned}
F^{\mu\nu}_R &= \partial_\mu r_\nu - \partial_\nu r_\mu - i\,[r_\mu, r_\nu] \\
&= \partial_\mu(v_\nu + a_\nu) - \partial_\nu(v_\mu + a_\mu) - i\,[(v_\mu + a_\mu)(v_\nu + a_\nu) - (v_\nu + a_\nu)(v_\mu + a_\mu)] \\
&= \partial_\mu v_\nu + \partial_\mu a_\nu - \partial_\nu v_\mu - \partial_\nu a_\mu - i\,(v_\mu v_\nu + v_\mu a_\nu + a_\mu v_\nu + a_\mu a_\nu - v_\nu v_\mu - v_\nu a_\mu - a_\nu v_\mu - a_\nu a_\mu) \\
&= (\partial_\mu v_\nu - \partial_\nu v_\mu) + (\partial_\mu a_\nu - \partial_\nu a_\mu) + i\,[v_\nu, v_\mu] + i\,[v_\nu, a_\mu] + i\,[a_\nu, v_\mu] + i\,[a_\nu, a_\mu]
\end{aligned}$$

$$\begin{aligned}
F^{\mu\nu}_L &= \partial_\mu l_\nu - \partial_\nu l_\mu - i\,[l_\mu, l_\nu] \\
&= \partial_\mu(v_\nu - a_\nu) - \partial_\nu(v_\mu - a_\mu) - i\,[(v_\mu - a_\mu)(v_\nu - a_\nu) - (v_\nu - a_\nu)(v_\mu - a_\mu)] \\
&= \partial_\mu v_\nu - \partial_\mu a_\nu - \partial_\nu v_\mu + \partial_\nu a_\mu - i\,(v_\mu v_\nu - v_\mu a_\nu - a_\mu v_\nu + a_\mu a_\nu - v_\nu v_\mu + v_\nu a_\mu + a_\nu v_\mu - a_\nu a_\mu) \\
&= (\partial_\mu v_\nu - \partial_\nu v_\mu) - (\partial_\mu a_\nu - \partial_\nu a_\mu) + i\,[v_\nu, v_\mu] - i\,[v_\nu, a_\mu] - i\,[a_\nu, v_\mu] + i\,[a_\nu, a_\mu]
\end{aligned}$$

the sum and the difference between them shown below

$$\begin{aligned}
F^{\mu\nu}_L + F^{\mu\nu}_R &= \partial_\mu l_\nu - \partial_\nu l_\mu - i\,[l_\mu, l_\nu] + \partial_\mu r_\nu - \partial_\nu r_\mu - i\,[r_\mu, r_\nu] \\
&= 2\partial_\mu(-e\mathscr{A}_\mu Q) - 2\partial_\nu(-e\mathscr{A}_\mu Q) + 2ieQ\,[\mathscr{A}_\mu, \mathscr{A}_\nu] \\
&= 2eQ\,\{(\partial_\nu \mathscr{A}_\mu - \partial_\mu \mathscr{A}_\nu) + i\,[\mathscr{A}_\mu, \mathscr{A}_\nu]\} \\
F^{\mu\nu}_L - F^{\mu\nu}_R &= 0
\end{aligned}$$

10.2 $f_\pm^{\mu\nu}$

$$f_\pm^{\mu\nu} = uF_L^{\mu\nu}u^\dagger \pm u^\dagger F_R^{\mu\nu}u$$

where u is the unitary square root of U, $u^2(x) = U(x)$ and $u(x) \to u'(x) = \sqrt{RUL^\dagger}$, with expansions

$$u(\phi) = \exp\left(\frac{i\phi}{\sqrt{2}F}\right) = 1 + \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2} + \cdots$$

$$u^\dagger(\phi) = \exp\left(\frac{-i\phi}{\sqrt{2}F}\right) = 1 - \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2} + \cdots$$

$$\begin{aligned}
f_+^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger + u^\dagger F_R^{\mu\nu}u \\
&\approx (1 + \frac{i\phi}{\sqrt{2}F})F_L^{\mu\nu}(1 - \frac{i\phi}{\sqrt{2}F}) + (1 - \frac{i\phi}{\sqrt{2}F})F_R^{\mu\nu}(1 + \frac{i\phi}{\sqrt{2}F}) \\
&= F_L^{\mu\nu} - F_L^{\mu\nu}\frac{i\phi}{\sqrt{2}F} + \frac{i\phi}{\sqrt{2}F}F_L^{\mu\nu} - \frac{i\phi}{\sqrt{2}F}F_L^{\mu\nu}\frac{i\phi}{\sqrt{2}F} + F_R^{\mu\nu} + F_R^{\mu\nu}\frac{i\phi}{\sqrt{2}F} - \frac{i\phi}{\sqrt{2}F}F_R^{\mu\nu} - \frac{i\phi}{\sqrt{2}F}F_R^{\mu\nu}\frac{i\phi}{\sqrt{2}F} \\
&= (F_L^{\mu\nu} + F_R^{\mu\nu}) - (F_L^{\mu\nu} - F_R^{\mu\nu})\frac{i\phi}{\sqrt{2}F} + \frac{i\phi}{\sqrt{2}F}(F_L^{\mu\nu} - F_R^{\mu\nu}) - \frac{i\phi}{\sqrt{2}F}(F_L^{\mu\nu} + F_R^{\mu\nu})\frac{i\phi}{\sqrt{2}F} \\
&= (F_L^{\mu\nu} + F_R^{\mu\nu}) - \frac{i\phi}{\sqrt{2}F}(F_L^{\mu\nu} + F_R^{\mu\nu})\frac{i\phi}{\sqrt{2}F} \\
&= 2eQ\left\{(\partial_\nu\mathscr{A}_\mu - \partial_\mu\mathscr{A}_\nu) + i\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]\right\} + \frac{e\phi Q\phi}{F}\left\{(\partial_\nu\mathscr{A}_\mu - \partial_\mu\mathscr{A}_\nu) + i\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]\right\} \\
&\approx -2eQ(\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu)
\end{aligned}$$

$$\begin{aligned}
f_-^{\mu\nu} &= uF_L^{\mu\nu}u^\dagger - u^\dagger F_R^{\mu\nu}u \\
&\approx (1 + \frac{i\phi}{\sqrt{2}F})F_L^{\mu\nu}(1 - \frac{i\phi}{\sqrt{2}F}) - (1 - \frac{i\phi}{\sqrt{2}F})F_R^{\mu\nu}(1 + \frac{i\phi}{\sqrt{2}F}) \\
&= (F_L^{\mu\nu} - F_R^{\mu\nu}) - (F_L^{\mu\nu} + F_R^{\mu\nu})\frac{i\phi}{\sqrt{2}F} + \frac{i\phi}{\sqrt{2}F}(F_L^{\mu\nu} + F_R^{\mu\nu}) - \frac{i\phi}{\sqrt{2}F}(F_L^{\mu\nu} - F_R^{\mu\nu})\frac{i\phi}{\sqrt{2}F} \\
&= \frac{i\phi}{\sqrt{2}F}2eQ\left\{(\partial_\nu\mathscr{A}_\mu - \partial_\mu\mathscr{A}_\nu) + i\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]\right\} - 2eQ\left\{(\partial_\nu\mathscr{A}_\mu - \partial_\mu\mathscr{A}_\nu) + i\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]\right\}\frac{i\phi}{\sqrt{2}F} \\
&= \frac{\sqrt{2}ie}{F}\left\{(\partial_\nu\mathscr{A}_\mu - \partial_\mu\mathscr{A}_\nu) + i\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]\right\}\left[\phi, Q\right] \\
&\approx -\frac{\sqrt{2}ie}{F}(\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu)\left[\phi, Q\right]
\end{aligned}$$

10.3 chi +/-

$$\chi = 2B_0(s+ip) = -\frac{2}{3F_0^2}\langle\bar{q}q\rangle_0(s+ip)$$
$$\chi_\pm = u^\dagger\chi u^\dagger \pm u\chi^\dagger u$$

Combine scalar fields and pseudo-scalar fields by $\chi \equiv 2B_0(s+ip)$. Easy to know $\chi+\chi^\dagger = 4B_0 s$; if $p=0$, then $\chi-\chi^\dagger = 4iB_0 p = 0$ or $\chi = \chi^\dagger$. Below we calculate $\chi_\pm$, with mass matrices shown in 3.4.2.

$$\begin{aligned}
\chi_+ &= u^\dagger\chi u^\dagger + u\chi^\dagger u \\
&\approx (1-\frac{i\phi}{\sqrt{2}F})\chi(1-\frac{i\phi}{\sqrt{2}F}) + (1+\frac{i\phi}{\sqrt{2}F})\chi^\dagger(1+\frac{i\phi}{\sqrt{2}F}) \\
&= \left(\chi - \frac{i\phi\chi}{\sqrt{2}F} - \frac{i\chi\phi}{\sqrt{2}F} - \frac{\phi\chi\phi}{2F^2}\right) + \left(\chi^\dagger + \frac{i\phi\chi^\dagger}{\sqrt{2}F} + \frac{i\chi^\dagger\phi}{\sqrt{2}F} - \frac{\phi\chi^\dagger\phi}{2F^2}\right) \\
&= (\chi+\chi^\dagger) - \frac{i\phi}{\sqrt{2}F}(\chi-\chi^\dagger) - (\chi-\chi^\dagger)\frac{i\phi}{\sqrt{2}F} - \frac{\phi(\chi+\chi^\dagger)\phi}{2F^2} \\
&= (\chi+\chi^\dagger) - \frac{\phi(\chi+\chi^\dagger)\phi}{2F^2} \\
&\approx (\chi+\chi^\dagger) = 4B_0 s = 4B_0\mathrm{diag}(m_u, m_d, m_s) \\
&= 4B_0\mathrm{diag}(\hat{m}, \hat{m}, m_s) = 2\mathrm{diag}(M_\pi^2, M_\pi^2, 2M_K^2 - M_\pi^2) \\
&= 2\begin{pmatrix} M_\pi^2 & 0 & 0 \\ 0 & M_\pi^2 & 0 \\ 0 & 0 & 2M_K^2 - M_\pi^2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\chi_- &= u^\dagger\chi u^\dagger - u\chi^\dagger u \\
&\approx (1-\frac{i\phi}{\sqrt{2}F})\chi(1-\frac{i\phi}{\sqrt{2}F}) - (1+\frac{i\phi}{\sqrt{2}F})\chi^\dagger(1+\frac{i\phi}{\sqrt{2}F}) \\
&= \left(\chi - \frac{i\phi\chi}{\sqrt{2}F} - \frac{i\chi\phi}{\sqrt{2}F} - \frac{\phi\chi\phi}{2F^2}\right) - \left(\chi^\dagger + \frac{i\phi\chi^\dagger}{\sqrt{2}F} + \frac{i\chi^\dagger\phi}{\sqrt{2}F} - \frac{\phi\chi^\dagger\phi}{2F^2}\right) \\
&= (\chi-\chi^\dagger) - \frac{i\phi}{\sqrt{2}F}(\chi+\chi^\dagger) - (\chi+\chi^\dagger)\frac{i\phi}{\sqrt{2}F} - \frac{\phi(\chi-\chi^\dagger)\phi}{2F^2} \\
&= -\frac{i\phi}{\sqrt{2}F}(\chi+\chi^\dagger) - (\chi+\chi^\dagger)\frac{i\phi}{\sqrt{2}F} \\
&= -\frac{\sqrt{2}i}{F}(\phi\chi + \chi\phi)
\end{aligned}$$

10.4 Chiral vielbein u_μ

Pay attention to u and u_μ, the latter is the so-called chiral vielbein.

$$u_\mu \equiv i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right]$$

under parity transformation, $u_\mu \to i\left[u(\partial^\mu - il^\mu)u^\dagger - u^\dagger(\partial^\mu - ir^\mu)u\right] = -u^\mu$ like axial vectors. Moreover, with external fields, *chiral vielbein* is odd-intrinsic-parity under external fields. $(\phi_i \to -\phi_i)$: $u_\mu = i\left[u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right] \to i\left[u\partial_\mu u^\dagger - u^\dagger\partial_\mu u\right] = -u_\mu$

$$\begin{aligned}
u_\mu &= i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right] \\
&\approx i\left\{\left(1-\frac{i\phi}{\sqrt{2}F}-\frac{\phi^2}{4F^2}\right)\left[\partial_\mu - i(-e\mathscr{A}_\mu Q)\right]\left(1+\frac{i\phi}{\sqrt{2}F}-\frac{\phi^2}{4F^2}\right)\right. \\
&\quad \left. -\left(1+\frac{i\phi}{\sqrt{2}F}-\frac{\phi^2}{4F^2}\right)\left[\partial_\mu - i(-e\mathscr{A}_\mu Q)\right]\left(1-\frac{i\phi}{\sqrt{2}F}-\frac{\phi^2}{4F^2}\right)\right\} \\
&= i\left\{\left(1-\frac{i\phi}{\sqrt{2}F}-\frac{\phi^2}{4F^2}\right)\left(\frac{i\partial_\mu\phi}{\sqrt{2}F}-\frac{(\partial_\mu\phi)\phi+\phi(\partial_\mu\phi)}{4F^2}+ie\mathscr{A}_\mu Q-\frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F}-\frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2}\right)\right. \\
&\quad \left. -\left(1+\frac{i\phi}{\sqrt{2}F}-\frac{\phi^2}{4F^2}\right)\left(-\frac{i\partial_\mu\phi}{\sqrt{2}F}-\frac{(\partial_\mu\phi)\phi+\phi(\partial_\mu\phi)}{4F^2}+ie\mathscr{A}_\mu Q+\frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F}-\frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2}\right)\right\} \\
&= i\left\{\left(1-\frac{i\phi}{\sqrt{2}F}-\frac{\phi^2}{4F^2}\right)\left(\frac{i\partial_\mu\phi}{\sqrt{2}F}-\frac{(\partial_\mu\phi)\phi+\phi(\partial_\mu\phi)}{4F^2}+ie\mathscr{A}_\mu Q-\frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F}-\frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2}\right)\right. \\
&\quad \left. +\left(1+\frac{i\phi}{\sqrt{2}F}-\frac{\phi^2}{4F^2}\right)\left(\frac{i\partial_\mu\phi}{\sqrt{2}F}+\frac{(\partial_\mu\phi)\phi+\phi(\partial_\mu\phi)}{4F^2}-ie\mathscr{A}_\mu Q-\frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F}+\frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2}\right)\right\} \\
&= 2i\left\{\frac{i\partial_\mu\phi}{\sqrt{2}F}-\frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F}+\frac{i\phi}{\sqrt{2}F}\cdot\frac{(\partial_\mu\phi)\phi+\phi(\partial_\mu\phi)}{4F^2}-\frac{i\phi}{\sqrt{2}F}\cdot ie\mathscr{A}_\mu Q+\frac{i\phi}{\sqrt{2}F}\cdot\frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2}\right. \\
&\quad \left. -\frac{\phi^2}{4F^2}\cdot\frac{i\partial_\mu\phi}{\sqrt{2}F}+\frac{\phi^2}{4F^2}\cdot\frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F}\right\} \\
&= -\frac{\sqrt{2}\partial_\mu\phi}{F}-\frac{\sqrt{2}ie\mathscr{A}_\mu Q\phi}{F}-\frac{\phi(\partial_\mu\phi)\phi+\phi^2(\partial_\mu\phi)}{2\sqrt{2}F^3}+\frac{\sqrt{2}ie\mathscr{A}_\mu\phi Q}{F}-\frac{i\phi e\mathscr{A}_\mu Q\phi^2}{2\sqrt{2}F^3} \\
&\quad +\frac{\phi^2\partial_\mu\phi}{2\sqrt{2}F^3}+\frac{ie\mathscr{A}_\mu\phi^2 Q\phi}{2\sqrt{2}F^3} \\
&= -\frac{\sqrt{2}\partial_\mu\phi}{F}+\frac{\sqrt{2}ie\mathscr{A}_\mu}{F}[\phi,Q]-\frac{\phi(\partial_\mu\phi)\phi}{2\sqrt{2}F^3}+\frac{ie\mathscr{A}_\mu}{2\sqrt{2}F^3}\left(\phi\,[Q,\phi]\,\phi\right)
\end{aligned}$$

For faster calculations, some terms vanish in line 4, according to their **difference** symbols:

	+/−	−/−	+/+	−/+	−/−
+/+	+/−	−/−	+/+	−/+	−/−
−/+	−/−	+/−	−/+	+/+	+/−
−/−	−/+	+/+	−/−	+/−	+/+

10.5 Chiral connection Γ_μ

Γ_μ is even-intrinsic-parity without external fields, as for pion fields.

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right] \to \frac{1}{2}\left[u\partial_\mu u^\dagger + u^\dagger\partial_\mu u\right] = \Gamma_\mu$$

$$\begin{aligned}
\Gamma_\mu &= \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right] \\
2\Gamma_\mu &= \left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\right] \\
&\approx \left\{\left(1 - \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right)\left[\partial_\mu - i(-e\mathscr{A}_\mu Q)\right]\left(1 + \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right)\right. \\
&\quad \left. + \left(1 + \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right)\left[\partial_\mu - i(-e\mathscr{A}_\mu Q)\right]\left(1 - \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right)\right\} \\
&= \left\{\left(1 - \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right)\left(\frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} + ie\mathscr{A}_\mu Q - \frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F} - \frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2}\right)\right. \\
&\quad \left. + \left(1 + \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right)\left(-\frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} + ie\mathscr{A}_\mu Q + \frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F} - \frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2}\right)\right\} \\
\Gamma_\mu &= -\frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} + ie\mathscr{A}_\mu Q - \frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2} - \frac{i\phi}{\sqrt{2}F}\cdot\frac{i\partial_\mu\phi}{\sqrt{2}F} + \frac{i\phi}{\sqrt{2}F}\cdot\frac{e\mathscr{A}_\mu Q\phi}{\sqrt{2}F} \\
&\quad + \frac{\phi^2}{4F^2}\cdot\frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} - \frac{\phi^2}{4F^2}\cdot ie\mathscr{A}_\mu Q + \frac{\phi^2}{4F^2}\cdot\frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2} \\
&= -\frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} + ie\mathscr{A}_\mu Q - \frac{ie\mathscr{A}_\mu Q\phi^2}{4F^2} + \frac{\phi\partial_\mu\phi}{2F^2} + \frac{ie\mathscr{A}_\mu\phi Q\phi}{2F^2} \\
&\quad + \frac{\phi^2(\partial_\mu\phi)\phi + \phi^3(\partial_\mu\phi)}{16F^4} - \frac{ie\mathscr{A}_\mu\phi^2 Q}{4F^2} + \frac{ie\mathscr{A}_\mu\phi^2 Q\phi^2}{16F^4} \\
&= \frac{\phi(\partial_\mu\phi) - (\partial_\mu\phi)\phi}{4F^2} + ie\mathscr{A}_\mu Q - \frac{ie\mathscr{A}_\mu}{4F^2}(Q\phi^2 + \phi^2 Q) + \frac{ie\mathscr{A}_\mu\phi Q\phi}{2F^2} \\
&\quad + \frac{\phi^2(\partial_\mu\phi)\phi + \phi^3(\partial_\mu\phi)}{16F^4} + \frac{ie\mathscr{A}_\mu\phi^2 Q\phi^2}{16F^4} \\
&\approx ie\mathscr{A}_\mu Q + \frac{\phi(\partial_\mu\phi) - (\partial_\mu\phi)\phi}{4F^2} - \frac{ie\mathscr{A}_\mu}{4F^2}(Q\phi^2 + \phi^2 Q) + \frac{ie\mathscr{A}_\mu\phi Q\phi}{2F^2}
\end{aligned}$$

For faster calculations, some terms vanish in line 4, according to their **same** symbols:

	+/−	−/−	+/+	−/+	−/−
+/+	+/−	−/−	+/+	−/+	−/−
−/+	−/−	+/−	−/+	+/+	+/−
−/−	−/+	+/+	−/−	+/−	+/+

10.6 $\Sigma_\mu^{L/R}$

$$
\begin{aligned}
\Sigma_\mu^L &= u^\dagger \partial_\mu u \\
&\approx \left(1 - \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right) \partial_\mu \left(1 + \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right) \\
&= \left(1 - \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right) \left(\frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2}\right) \\
&= \frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} - \frac{i\phi}{\sqrt{2}F} \cdot \frac{i\partial_\mu\phi}{\sqrt{2}F} + \frac{i\phi}{\sqrt{2}F} \cdot \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} \\
&\quad - \frac{\phi^2}{4F^2} \cdot \frac{i\partial_\mu\phi}{\sqrt{2}F} + \frac{\phi^2}{4F^2} \cdot \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} \\
&= \frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} + \frac{\phi\partial_\mu\phi}{2F^2} + \frac{i(\phi(\partial_\mu\phi)\phi + \phi^2(\partial_\mu\phi))}{4\sqrt{2}F^3} \\
&\quad - \frac{i\phi^2\partial_\mu\phi}{4\sqrt{2}F^3} + \frac{\phi^2(\partial_\mu\phi)\phi + \phi^3(\partial_\mu\phi)}{16F^4} \\
&= \frac{i\partial_\mu\phi}{\sqrt{2}F} + \frac{\phi(\partial_\mu\phi) - (\partial_\mu\phi)\phi}{4F^2} + \frac{i\phi(\partial_\mu\phi)\phi}{4\sqrt{2}F^3} + \frac{\phi^2(\partial_\mu\phi)\phi + \phi^3(\partial_\mu\phi)}{16F^4}
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_\mu^R &= u \partial_\mu u^\dagger \\
&\approx \left(1 + \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right) \partial_\mu \left(1 - \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right) \\
&= \left(1 + \frac{i\phi}{\sqrt{2}F} - \frac{\phi^2}{4F^2}\right) \left(-\frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2}\right) \\
&= -\frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} - \frac{i\phi}{\sqrt{2}F} \cdot \frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{i\phi}{\sqrt{2}F} \cdot \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} \\
&\quad + \frac{\phi^2}{4F^2} \cdot \frac{i\partial_\mu\phi}{\sqrt{2}F} + \frac{\phi^2}{4F^2} \cdot \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} \\
&= -\frac{i\partial_\mu\phi}{\sqrt{2}F} - \frac{(\partial_\mu\phi)\phi + \phi(\partial_\mu\phi)}{4F^2} + \frac{\phi\partial_\mu\phi}{2F^2} - \frac{i(\phi(\partial_\mu\phi)\phi + \phi^2(\partial_\mu\phi))}{4\sqrt{2}F^3} \\
&\quad + \frac{i\phi^2\partial_\mu\phi}{4\sqrt{2}F^3} + \frac{\phi^2(\partial_\mu\phi)\phi + \phi^3(\partial_\mu\phi)}{16F^4} \\
&= -\frac{i\partial_\mu\phi}{\sqrt{2}F} + \frac{\phi(\partial_\mu\phi) - (\partial_\mu\phi)\phi}{4F^2} - \frac{i\phi(\partial_\mu\phi)\phi}{4\sqrt{2}F^3} + \frac{\phi^2(\partial_\mu\phi)\phi + \phi^3(\partial_\mu\phi)}{16F^4}
\end{aligned}
$$

10.7 Covariant derivative & $h^{\pm}_{\mu\nu}$

$\nabla_\mu V = \partial_\mu V + [\Gamma_\mu, V]$. Approximation: $\Gamma_\alpha \approx ie\mathscr{A}_\alpha Q$ as well as

$$u_\mu \approx -\frac{\sqrt{2}\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\mathscr{A}_\mu}{F}[\phi, Q]$$

$$\begin{aligned}
\nabla_\alpha u_\mu &= \partial_\alpha u_\mu + [\Gamma_\alpha, u_\mu] \\
&= \partial_\alpha\left\{-\frac{\sqrt{2}\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\mathscr{A}_\mu}{F}[\phi, Q]\right\} + \left[ie\mathscr{A}_\alpha Q, -\frac{\sqrt{2}\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\mathscr{A}_\mu}{F}[\phi, Q]\right] \\
&\approx \partial_\alpha\left\{-\frac{\sqrt{2}\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\mathscr{A}_\mu}{F}[\phi, Q]\right\} + \left[ie\mathscr{A}_\alpha Q, -\frac{\sqrt{2}\partial_\mu\phi}{F}\right] \\
&= -\frac{\sqrt{2}\partial_\alpha\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\partial_\alpha\mathscr{A}_\mu}{F}[\phi, Q] + \frac{\sqrt{2}ie\mathscr{A}_\mu}{F}[\partial_\mu\phi, Q] - ie\mathscr{A}_\alpha Q\cdot\frac{\sqrt{2}\partial_\mu\phi}{F} + \frac{\sqrt{2}\partial_\mu\phi}{F}\cdot ie\mathscr{A}_\alpha Q \\
&= -\frac{\sqrt{2}\partial_\alpha\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\partial_\alpha\mathscr{A}_\mu}{F}[\phi, Q] + \frac{\sqrt{2}ie\mathscr{A}_\mu}{F}[\partial_\alpha\phi, Q] + \frac{\sqrt{2}ie\mathscr{A}_\alpha}{F}[\partial_\mu\phi, Q] \\
&= -\frac{\sqrt{2}\partial_\alpha\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\partial_\alpha\mathscr{A}_\mu}{F}[\phi, Q] + \frac{2\sqrt{2}ie\mathscr{A}_\alpha}{F}[\partial_\mu\phi, Q]
\end{aligned}$$

$$h^{\pm}_{\mu\nu} = \nabla_\mu u_\nu \pm \nabla_\nu u_\mu$$

$$\begin{aligned}
h^{-}_{\mu\nu} &= \nabla_\mu u_\nu - \nabla_\nu u_\mu \\
&= \left(-\frac{\sqrt{2}\partial_\mu\partial_\nu\phi}{F} + \frac{\sqrt{2}ie\partial_\mu\mathscr{A}_\nu}{F}[\phi, Q] + \frac{2\sqrt{2}ie\mathscr{A}_\mu}{F}[\partial_\nu\phi, Q]\right) \\
&\quad - \left(-\frac{\sqrt{2}\partial_\nu\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\partial_\nu\mathscr{A}_\mu}{F}[\phi, Q] + \frac{2\sqrt{2}ie\mathscr{A}_\nu}{F}[\partial_\mu\phi, Q]\right) \\
&= \frac{\sqrt{2}}{F}(\partial_\nu\partial_\mu - \partial_\mu\partial_\nu)Q + \frac{\sqrt{2}ie}{F}[\phi, Q](\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu) + \frac{2\sqrt{2}ie}{F}(\mathscr{A}_\mu[\partial_\nu\phi, Q] - \mathscr{A}_\nu[\partial_\mu\phi, Q]) \\
&= \frac{\sqrt{2}ie}{F}[\phi, Q](\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu) \\
&\cong -f_-^{\mu\nu}
\end{aligned}$$

$$\begin{aligned}
h^{+}_{\mu\nu} &= \nabla_\mu u_\nu + \nabla_\nu u_\mu \\
&= \left(-\frac{\sqrt{2}\partial_\mu\partial_\nu\phi}{F} + \frac{\sqrt{2}ie\partial_\mu\mathscr{A}_\nu}{F}[\phi, Q] + \frac{2\sqrt{2}ie\mathscr{A}_\mu}{F}[\partial_\nu\phi, Q]\right) \\
&\quad + \left(-\frac{\sqrt{2}\partial_\nu\partial_\mu\phi}{F} + \frac{\sqrt{2}ie\partial_\nu\mathscr{A}_\mu}{F}[\phi, Q] + \frac{2\sqrt{2}ie\mathscr{A}_\nu}{F}[\partial_\mu\phi, Q]\right) \\
&= -\frac{\sqrt{2}}{F}(\partial_\nu\partial_\mu + \partial_\mu\partial_\nu)\phi + \frac{\sqrt{2}ie}{F}[\phi, Q](\partial_\mu\mathscr{A}_\nu + \partial_\nu\mathscr{A}_\mu) + \frac{2\sqrt{2}ie}{F}(\mathscr{A}_\mu[\partial_\nu\phi, Q] + \mathscr{A}_\nu[\partial_\mu\phi, Q])
\end{aligned}$$

11 How to calculate

Ref "Odd–intrinsic–parity processes within the Resonance Effective Theory of QCD", P. D. Ruiz-Femenía, A. Pich and J. Portolés.

11.1 Lagrangians

The low-energy behavior of QCD for the light quark sector (u, d, s) is known to be ruled by the spontaneous breaking of chiral symmetry giving rise to the lightest hadron degrees of freedom, identified with the octet of pseudo-scalar mesons. The corresponding effective realization of QCD describing the interaction between the Goldstone fields is called chiral perturbation theory [1,2,3]. The effective Lagrangian to lowest order in derivatives, $\mathcal{O}(p^2)$, is given by :

$$\mathcal{L}_\chi^{(2)} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle$$

with parameters mentioned previously

$$u_\mu = i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right]$$

$$\chi_+ = u^\dagger \chi u^\dagger + u\chi^\dagger u$$

Unitary matrix in the flavor space:

$$u(\phi) = \exp\left(\frac{i\Phi}{\sqrt{2}F}\right)$$

is a non-linear parameterization of the Goldstone octet of fields:

$$\Phi(x) \equiv \frac{\vec{\lambda}}{\sqrt{2}}\vec{\phi} = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}$$

The external hermitian matrix fields r_μ, l_μ, s, p promote the global SU(3)$_R$×SU(3)$_L$ symmetry of the lagrangian to a local one, and generate Green functions of quark currents by taking appropriate functional derivatives. Interactions with electroweak bosons can be accommodated through the vector $v_\mu = (r_\mu + l_\mu)/2$ and axial-vector $a_\mu = (r_\mu - l_\mu)/2$ fields, while the scalar field $s = \mathcal{M} + \cdots$ provides a very convenient way of incorporating explicit chiral symmetry breaking through the quark masses.

The generating functional $Z[v, a, s, p]$ calculated in terms of the external sources is manifestly chiral invariant, but the physically interesting Green functions (with broken chiral symmetry) are obtained by taking a particular direction in flavor space through functional differentiation. Finally, the lagrangian $\mathcal{L}_\chi^{(2)}$ is settled by fixing F and B_0 from the phenomenology: $F \simeq F_\pi \simeq 92.4$ MeV is the decay constant of the charged pion and $B_0F^2 = -\langle 0|\bar{\psi}\psi|0\rangle_0$ in the chiral limit.

$$\mathcal{L}_{\rm V} = \mathcal{L}_{\rm Kin}(V) + \mathcal{L}_2(V)$$

$$\mathcal{L}_{\text{Kin}}(V) = -\frac{1}{2}\langle \nabla^\lambda V_{\lambda\mu} \nabla_\nu V^{\nu\mu} - \frac{M_V^2}{2} V_{\mu\nu} V^{\mu\nu} \rangle$$

M_V is the mass of the lowest octet of vector resonances under SU(3).

$$\nabla_\mu V = \partial_\mu V + [\Gamma_\mu, V]$$

$$\Gamma_\mu = \frac{1}{2}\left[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger \right]$$

in such a way that $\nabla_\mu V$ also transforms as an octet under the action of the group. For the interaction lagrangian $\mathcal{L}_2(V)$ we have

$$\mathcal{L}_2(V) = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu} u^\mu u^\nu \rangle$$

$$f_\pm^{\mu\nu} = u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u$$

where $F_{L/R}^{\mu\nu}$ is the field strength tensor of left/right external field $l_\mu, r\mu$, and F_V, G_V are real couplings. The octet fields are written in the usual matrix notation

$$V_{\mu\nu} = \frac{\vec{\lambda}}{\sqrt{2}}\vec{V}_{\mu\nu} = \begin{pmatrix} \frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 & \rho^+ & K^{*+} \\ \rho^- & -\frac{1}{\sqrt{2}}\rho^0 + \frac{1}{\sqrt{6}}\omega_8 & K^{*0} \\ K^{*-} & \bar{K}^{*0} & -\frac{2}{\sqrt{6}}\omega_8 \end{pmatrix}_{\mu\nu}$$

the effective Lagrangian $\mathcal{L}_{\chi\text{V}} \equiv \mathcal{L}_\chi^{(2)} + \mathcal{L}_\text{V}$ is enough to satisfy the short-distance QCD constraints where vector resonances play an important role.

$$\mathcal{L}_V^{\text{odd}} = \mathcal{L}_{\text{VJP}} + \mathcal{L}_{\text{VVP}} = \sum_{a=1}^{7} \frac{c_a}{M_V}\mathcal{O}_{\text{VJP}}^a + \sum_{a=1}^{4} d_a \mathcal{O}_{\text{VVP}}^a$$

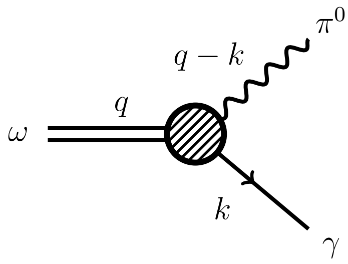
11.2 VJP: vector, external, pseudoscalar

VJP terms, vertices with one vector resonance and one external vector source plus one pseudoscalar. Herein, ω is the vector resonance, γ is external vector source, and π is pseudoscalar.

$$\begin{aligned}
\mathcal{O}_{\text{VJP}}^1 &= \epsilon_{\mu\nu\rho\sigma}\langle \{V^{\mu\nu}, f_+^{\rho\alpha}\}\, \nabla_\alpha u^\sigma \rangle & \mathcal{O}_{\text{VJP}}^2 &= \epsilon_{\mu\nu\rho\sigma}\langle \{V^{\mu\alpha}, f_+^{\rho\sigma}\}\, \nabla_\alpha u^\nu \rangle \\
\mathcal{O}_{\text{VJP}}^3 &= i\epsilon_{\mu\nu\rho\sigma}\langle \{V^{\mu\nu}, f_+^{\rho\sigma}\}\, \chi_- \rangle & \mathcal{O}_{\text{VJP}}^4 &= i\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}\, [f_-^{\rho\sigma}, \chi_+] \rangle \\
\mathcal{O}_{\text{VJP}}^5 &= \epsilon_{\mu\nu\rho\sigma}\langle \{\nabla_\alpha V^{\mu\nu}, f_+^{\rho\alpha}\}\, u^\sigma \rangle & \mathcal{O}_{\text{VJP}}^6 &= \epsilon_{\mu\nu\rho\sigma}\langle \{\nabla_\alpha V^{\mu\alpha}, f_+^{\rho\sigma}\}\, u^\nu \rangle \\
\mathcal{O}_{\text{VJP}}^7 &= \epsilon_{\mu\nu\rho\sigma}\langle \{\nabla^\sigma V^{\mu\nu}, f_+^{\rho\alpha}\}\, u_\alpha \rangle
\end{aligned}$$

Revise Levi-Civita symbol, such as $\epsilon_{1234} = +1, \epsilon_{2134} = -1$. After odd-time permutation, it changes the symbol.Here we prove $\epsilon_{\mu\nu\rho\sigma} q^\mu q^\sigma = 0$ for the later convenience.

$$\epsilon_{\mu\nu\rho\sigma} q^\mu q^\sigma = -\epsilon_{\sigma\nu\rho\mu} q^\mu q^\sigma = -\epsilon_{\mu\nu\rho\sigma} q^\sigma q^\mu$$



For massless photon, $k^2 = k_\mu k^\mu = 0$. As for the process $\omega \to \gamma\pi$, the product of two 4-vectors:

$$(q-k)_\alpha k^\alpha = q_\alpha k^\alpha - k_\alpha k^\alpha = q \cdot k = \frac{(q^2+k^2)-(q-k)^2}{2} = \frac{M_\omega^2 - M_\pi^2}{2}$$

11.2.1 $\mathcal{O}^1_{\text{VJP}}$

$$\begin{aligned}
\mathcal{O}^1_{\text{VJP}} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\alpha}\}\,\nabla_\alpha u^\sigma\rangle \\
&= \epsilon_{\mu\nu\rho\sigma}\langle\left\{V^{\mu\nu}, uF_L^{\rho\alpha}u^\dagger + u^\dagger F_R^{\rho\alpha}u\right\}\nabla_\alpha i\left[u^\dagger(\partial^\sigma - ir^\sigma)u - u(\partial^\sigma - il^\sigma)u^\dagger\right]\rangle \\
&\approx \epsilon_{\mu\nu\rho\sigma}\left\langle \{V^{\mu\nu}, 2eQ(\partial^\alpha\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\alpha)\}\,\nabla_\alpha\left(-\frac{\sqrt{2}\partial^\sigma\phi}{F}\right)\right\rangle \\
&\approx \epsilon_{\mu\nu\rho\sigma}\left\langle \{V^{\mu\nu}, Q\}\,\frac{2\sqrt{2}e}{F}(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\partial_\alpha\partial^\sigma\phi\right\rangle
\end{aligned}$$

$$\begin{aligned}
\mathcal{L} \supset \frac{C_1}{M_V}\mathcal{O}^1_{\text{VJP}} &= \frac{2\sqrt{2}eC_1}{M_V F}\epsilon_{\mu\nu\rho\sigma}\left\langle\{V^{\mu\nu}, Q\}\,\partial_\alpha\partial^\sigma\phi\right\rangle(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho) \\
&= \frac{2\sqrt{2}eC_1}{M_V F}\epsilon_{\mu\nu\rho\sigma}\omega^{\mu\nu}\partial_\alpha\partial^\sigma\pi^0(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)
\end{aligned}$$

Attention to the imaginary unit before Lagrangians

$$\begin{aligned}
i\mathcal{M}^{(1)}_{\text{VJP}} &= \frac{\delta^3 i\mathcal{L}^{(1)}}{\delta A\delta\pi^0\delta\omega} = \frac{\delta^3 i}{\delta A\delta\pi^0\delta\omega}\frac{C_1}{M_V}\mathcal{O}^1_{\text{VJP}} \\
&= \frac{\delta^3 i}{\delta A\delta\pi^0\delta\omega}\left[\frac{2\sqrt{2}eC_1}{M_V F}\epsilon_{\mu\nu\rho\sigma}\omega^{\mu\nu}\partial_\alpha\partial^\sigma\pi^0(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\right] \\
&= \frac{2\sqrt{2}ieC_1}{M_V F}\epsilon_{\mu\nu\rho\sigma}\left[\frac{1}{M_\omega}(-iq^\mu\epsilon_\omega^\nu + iq^\nu\epsilon_\omega^\mu)\cdot i(q-k)_\alpha\cdot i(q-k)^\sigma\cdot(ik^\rho\epsilon_\gamma^\alpha - ik^\alpha\epsilon_\gamma^\rho)\right] \\
&= \frac{2\sqrt{2}ieC_1}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}(-q^\mu\epsilon_\omega^\nu + q^\nu\epsilon_\omega^\mu)(q-k)_\alpha(-k)^\sigma(k^\rho\epsilon_\gamma^\alpha - k^\alpha\epsilon_\gamma^\rho) \\
&= \frac{2\sqrt{2}ieC_1}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}(-q^\mu\epsilon_\omega^\nu + q^\nu\epsilon_\omega^\mu)(q-k)_\alpha(-k)^\sigma(-k^\alpha\epsilon_\gamma^\rho) \\
&= -\frac{2\sqrt{2}ieC_1}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\cdot 2q^\mu\epsilon_\omega^\nu(q-k)_\alpha k^\sigma k^\alpha\epsilon_\gamma^\rho = -\frac{4\sqrt{2}ieC_1}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}q^\mu\epsilon_\omega^\nu k^\sigma\epsilon_\gamma^\rho(q_\alpha\cdot k^\alpha) \\
&= +\frac{4\sqrt{2}ieC_1}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon_\gamma^\mu\epsilon_\omega^\nu q^\rho k^\sigma(M_\omega^2 - M_\pi^2) = -\frac{4\sqrt{2}ieC_1}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon_\omega^\mu\epsilon_\gamma^\nu q^\rho k^\sigma(M_\omega^2 - M_\pi^2)
\end{aligned}$$

11.2.2 $\mathcal{O}^2_{\text{VJP}}$

$$
\begin{aligned}
\mathcal{O}^2_{\text{VJP}} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\alpha}, f_+^{\rho\sigma}\}\,\nabla_\alpha u^\nu\rangle \\
\mathcal{L} \supset \frac{C_2}{M_V}\mathcal{O}^2_{\text{VJP}} &= \frac{C_2}{M_V}\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\alpha}, f_+^{\rho\sigma}\}\,\nabla_\alpha u^\nu\rangle \\
&\approx \frac{C_2}{M_V}\epsilon_{\mu\nu\rho\sigma}\left\langle \{V^{\mu\alpha}, 2eQ(\partial^\sigma\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\sigma)\}\left(-\frac{\sqrt{2}}{F}\partial_\alpha\partial^\nu\phi\right)\right\rangle \\
&= \frac{2\sqrt{2}eC_2}{M_V F}\epsilon_{\mu\nu\rho\sigma}\left\langle\{V^{\mu\alpha}, Q\}\,(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\partial_\alpha\partial^\nu\phi\right\rangle \\
&= \frac{2\sqrt{2}eC_2}{M_V F}\epsilon_{\mu\nu\rho\sigma}\omega^{\mu\alpha}\partial_\alpha\partial^\nu\pi^0(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}^{(2)}_{\text{VJP}} &= \frac{\delta^3 i\mathcal{L}^{(2)}}{\delta A\delta\pi^0\delta\omega} = \frac{\delta^3 i}{\delta A\delta\pi^0\delta\omega}\frac{C_2}{M_V}\mathcal{O}^2_{\text{VJP}} \\
&= \frac{2\sqrt{2}ieC_2}{M_V F}\epsilon_{\mu\nu\rho\sigma}\frac{1}{M_\omega}(-iq^\mu\epsilon_\omega^\alpha + iq^\alpha\epsilon_\omega^\mu)\cdot i(q-k)_\alpha\cdot i(q-k)^\nu\cdot(ik^\rho\epsilon_\gamma^\sigma - ik^\sigma\epsilon_\gamma^\rho) \\
&= \frac{2\sqrt{2}ieC_2}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}q^\alpha\epsilon_\omega^\mu\cdot(q-k)_\alpha\cdot q^\nu\cdot(k^\rho\epsilon_\gamma^\sigma - k^\sigma\epsilon_\gamma^\rho) \\
&= \frac{2\sqrt{2}ieC_2}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\cdot\epsilon_\omega^\mu q^\nu\cdot 2k^\rho\epsilon_\gamma^\sigma\cdot(q^\alpha q_\alpha - q^\alpha\cdot k_\alpha) \\
&= \frac{2\sqrt{2}ieC_2}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\cdot\epsilon_\omega^\mu q^\nu\cdot 2k^\rho\epsilon_\gamma^\sigma\left(M_\omega^2 - \frac{M_\omega^2 - M_\pi^2}{2}\right) \\
&= \frac{2\sqrt{2}ieC_2}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon_\omega^\mu\epsilon_\gamma^\nu q^\rho k^\sigma\left(M_\omega^2 + M_\pi^2\right)
\end{aligned}
$$

11.2.3 $\mathcal{O}^3_{\text{VJP}}$

$$
\begin{aligned}
\mathcal{O}^3_{\text{VJP}} &= i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\sigma}\}\,\chi_-\rangle \\
\mathcal{L} \supset \frac{C_3}{M_V}\mathcal{O}^3_{\text{VJP}} &= \frac{C_3}{M_V}i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\sigma}\}\,\chi_-\rangle \\
&\approx \frac{iC_3}{M_V}\epsilon_{\mu\nu\rho\sigma}\left\langle\{V^{\mu\nu}, 2eQ(\partial^\sigma\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\sigma)\}\left(-\frac{\sqrt{2}i}{F}(\chi\phi + \phi\chi)\right)\right\rangle \\
&= \frac{2\sqrt{2}eC_3}{M_V F}\epsilon_{\mu\nu\rho\sigma}\omega^{\mu\nu}(\partial^\sigma\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\sigma)\cdot 2M_\pi^2\pi^0
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}^{(3)}_{\mathrm{VJP}} &= \frac{\delta^3 i\mathcal{L}^{(3)}}{\delta A\delta\pi^0\delta\omega} = \frac{\delta^3 i}{\delta A\delta\pi^0\delta\omega}\frac{C_3}{M_V}\mathcal{O}^3_{\mathrm{VJP}} \\
&= \frac{\delta^3 i}{\delta A\delta\pi^0\delta\omega}\left(\frac{2\sqrt{2}eC_3}{M_V F}\epsilon_{\mu\nu\rho\sigma}\omega^{\mu\nu}(\partial^\sigma\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\sigma)\cdot 2M_\pi^2\pi^0\right) \\
&= \frac{2\sqrt{2}ieC_3}{M_V F}\epsilon_{\mu\nu\rho\sigma}\frac{1}{M_\omega}(-iq^\mu\epsilon^\nu_\omega + iq^\nu\epsilon^\mu_\omega)\cdot 2M_\pi^2\cdot(ik^\sigma\epsilon^\rho_\gamma - ik^\rho\epsilon^\sigma_\gamma) \\
&= \frac{2\sqrt{2}ieC_3}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\cdot 2q^\mu\epsilon^\nu_\omega\cdot 2M_\pi^2\cdot 2k^\sigma\epsilon^\rho_\gamma \\
&= \frac{2\sqrt{2}ieC_3}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\nu_\omega\epsilon^\rho_\gamma q^\mu k^\sigma\cdot 8M_\pi^2 \\
&= \frac{2\sqrt{2}ieC_3}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_\gamma q^\rho k^\sigma\cdot 8M_\pi^2
\end{aligned}
$$

11.2.4 $\mathcal{O}^4_{\mathrm{VJP}}$

$$
\begin{aligned}
\mathcal{O}^4_{\mathrm{VJP}} &= i\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}\left[f_-^{\rho\sigma},\chi_+\right]\rangle \\
&\approx i\epsilon_{\mu\nu\rho\sigma}\left\langle V^{\mu\nu}\cdot\left[\frac{\sqrt{2}ie}{F}(\partial^\sigma\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\sigma)\left[\phi,Q\right],2\chi\right]\right\rangle \\
&= \frac{\sqrt{2}e}{F}\epsilon_{\mu\nu\rho\sigma}\left\langle V^{\mu\nu}\cdot[(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)(\phi Q - Q\phi),2\chi]\right\rangle \\
&= \frac{2\sqrt{2}e}{F}\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}\left[(\phi Q - Q\phi),\chi\right]\rangle(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho) = 0
\end{aligned}
$$

11.2.5 $\mathcal{O}^5_{\mathrm{VJP}}$

$$
\begin{aligned}
\mathcal{O}^5_{\mathrm{VJP}} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, f_+^{\rho\alpha}\}\,u^\sigma\rangle \\
\mathcal{L}\supset\frac{C_5}{M_V}\mathcal{O}^5_{\mathrm{VJP}} &= \frac{C_5}{M_V}\epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, f_+^{\rho\alpha}\}\,u^\sigma\rangle \\
&\approx \frac{C_5}{M_V}\epsilon_{\mu\nu\rho\sigma}\left\langle\{\partial_\alpha V^{\mu\nu}, 2eQ(\partial^\alpha\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\alpha)\}\left(-\frac{\sqrt{2}}{F}\partial^\sigma\phi\right)\right\rangle \\
&= \frac{2\sqrt{2}eC_5}{M_V F}\epsilon_{\mu\nu\rho\sigma}\langle\{\partial_\alpha V^{\mu\nu}, Q\}\,(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\partial^\sigma\phi\rangle \\
&= \frac{2\sqrt{2}eC_5}{M_V F}\epsilon_{\mu\nu\rho\sigma}\partial_\alpha\omega^{\mu\nu}\partial^\sigma\pi^0(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}^{(5)}_{\text{VJP}} &= \frac{\delta^3 i\mathcal{L}^{(5)}}{\delta A\delta\pi^0\delta\omega} = \frac{\delta^3 i}{\delta A\delta\pi^0\delta\omega}\frac{C_5}{M_V}\mathcal{O}^5_{\text{VJP}} \\
&= \frac{2\sqrt{2}ieC_5}{M_V F}\epsilon_{\mu\nu\rho\sigma}\cdot\frac{1}{M_\omega}\cdot(-i)q_\alpha\cdot(-iq^\mu\epsilon^\nu_\omega + iq^\nu\epsilon^\mu_\omega)\cdot i(q-k)^\sigma\cdot(ik^\rho\epsilon^\alpha_\gamma - ik^\alpha\epsilon^\rho_\gamma) \\
&= \frac{2\sqrt{2}ieC_5}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\cdot q_\alpha\cdot(q^\mu\epsilon^\nu_\omega - q^\nu\epsilon^\mu_\omega)\cdot(q-k)^\sigma\cdot(k^\rho\epsilon^\alpha_\gamma - k^\alpha\epsilon^\rho_\gamma) \\
&= \frac{2\sqrt{2}ieC_5}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\cdot q_\alpha\cdot(q^\mu\epsilon^\nu_\omega - q^\nu\epsilon^\mu_\omega)\cdot(-k)^\sigma\cdot(k^\rho\epsilon^\alpha_\gamma - k^\alpha\epsilon^\rho_\gamma) \\
&= \frac{2\sqrt{2}ieC_5}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\cdot 2q^\nu\epsilon^\mu_\omega\cdot(-k)^\sigma\cdot q_\alpha\cdot k^\alpha\epsilon^\rho_\gamma \\
&= \frac{2\sqrt{2}ieC_5}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\cdot 2q^\nu\epsilon^\mu_\omega\epsilon^\rho_\gamma\cdot(-k)^\sigma\cdot\left(\frac{M^2_\omega - M^2_\pi}{2}\right) \\
&= \frac{2\sqrt{2}ieC_5}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\rho_\gamma q^\sigma k^\nu(M^2_\omega - M^2_\pi) = \frac{2\sqrt{2}ieC_5}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_\gamma q^\rho k^\sigma(M^2_\omega - M^2_\pi)
\end{aligned}
$$

11.2.6 $\mathcal{O}^6_{\text{VJP}}$

$$
\begin{aligned}
\mathcal{O}^6_{\text{VJP}} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\alpha}, f^{\rho\sigma}_+\}\,u^\nu\rangle \\
\mathcal{L} \supset \frac{C_6}{M_V}\mathcal{O}^6_{\text{VJP}} &= \frac{C_6}{M_V}\epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\alpha}, f^{\rho\sigma}_+\}\,u^\nu\rangle \\
&\approx \frac{C_6}{M_V}\epsilon_{\mu\nu\rho\sigma}\left\langle\{\partial_\alpha V^{\mu\alpha}, 2eQ(\partial^\sigma\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\sigma)\}\left(-\frac{\sqrt{2}}{F}\partial^\nu\phi\right)\right\rangle \\
&= \frac{2\sqrt{2}eC_6}{M_V F}\epsilon_{\mu\nu\rho\sigma}\langle\{\partial_\alpha V^{\mu\alpha}, Q\}\,\partial^\nu\phi\rangle(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho) \\
&= \frac{2\sqrt{2}eC_6}{M_V F}\epsilon_{\mu\nu\rho\sigma}\partial_\alpha\omega^{\mu\alpha}\partial^\nu\pi^0(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}^{(6)}_{\text{VJP}} &= \frac{\delta^3 i\mathcal{L}^{(6)}}{\delta A\delta\pi^0\delta\omega} = \frac{\delta^3 i}{\delta A\delta\pi^0\delta\omega}\frac{C_6}{M_V}\mathcal{O}^6_{\text{VJP}} \\
&= \frac{\delta^3 i}{\delta A\delta\pi^0\delta\omega}\left[\frac{2\sqrt{2}eC_6}{M_V F}\epsilon_{\mu\nu\rho\sigma}\partial_\alpha\omega^{\mu\alpha}\partial^\nu\pi^0(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\right] \\
&= \frac{2\sqrt{2}ieC_6}{M_V F}\epsilon_{\mu\nu\rho\sigma}\cdot(-i)q_\alpha\cdot(-iq^\mu\epsilon^\alpha_\omega + iq^\alpha\epsilon^\mu_\omega)\cdot i(q-k)^\nu\cdot(ik^\rho\epsilon^\sigma_\gamma - ik^\sigma\epsilon^\rho_\gamma) \\
&= \frac{2\sqrt{2}ieC_6}{M_V F}\epsilon_{\mu\nu\rho\sigma}\cdot q_\alpha\cdot(q^\alpha\epsilon^\mu_\omega - q^\mu\epsilon^\alpha_\omega)\cdot q^\nu\cdot(k^\sigma\epsilon^\rho_\gamma - k^\rho\epsilon^\sigma_\gamma) \\
&= \frac{2\sqrt{2}ieC_6}{M_V F}\epsilon_{\mu\nu\rho\sigma}\cdot q_\alpha\cdot q^\alpha\epsilon^\mu_\omega\cdot q^\nu\cdot 2k^\sigma\epsilon^\rho_\gamma \\
&= \frac{4\sqrt{2}ieC_6}{M_V F}\epsilon_{\mu\nu\rho\sigma}M^2_\omega\cdot\epsilon^\mu_\omega\epsilon^\rho_\gamma q^\nu k^\sigma = -\frac{4\sqrt{2}ieC_6}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}M^2_\omega\cdot\epsilon^\mu_\omega\epsilon^\nu_\gamma q^\rho k^\sigma
\end{aligned}
$$

11.2.7 $\mathcal{O}^7_{\text{VJP}}$

$$\begin{aligned}
\frac{C_7}{M_V}\mathcal{O}^7_{\text{VJP}} &= \frac{C_7}{M_V}\epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, f_+^{\rho\alpha}\}\, u_\alpha\rangle \\
&\approx \frac{C_7}{M_V}\epsilon_{\mu\nu\rho\sigma}\left\langle\{\partial^\sigma V^{\mu\nu}, 2eQ(\partial^\alpha\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\sigma)\}\left(-\frac{\sqrt{2}}{F}\partial_\alpha\phi\right)\right\rangle \\
&= \frac{2\sqrt{2}eC_7}{M_V F}\epsilon_{\mu\nu\rho\sigma}\langle\{\partial^\sigma V^{\mu\nu}, Q\}\,\partial_\alpha\phi\rangle(\partial^\rho\mathscr{A}^\sigma - \partial^\alpha\mathscr{A}^\rho) = 0
\end{aligned}$$

$$\epsilon_{\mu\nu\rho\sigma}\partial^\sigma V^{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\partial^\sigma(\partial^\mu V^\nu + \partial^\nu V^\mu) = (\epsilon_{\mu\nu\rho\sigma} + \epsilon_{\nu\mu\rho\sigma})\partial^\sigma\partial^\mu V^\nu = 0$$

11.2.8 Scattering amplitude of $\mathcal{L}_{\text{VJP}}$

In summary, total Lagrangian of VJP process is the sum of each term.

$$\mathcal{L}_{\text{VJP}} = \sum_{a=1}^{7}\frac{c_a}{M_V}\mathcal{O}^a_{\text{VJP}}$$

And the total scattering amplitude of $\omega\to\gamma\pi$ as the sum of each term.

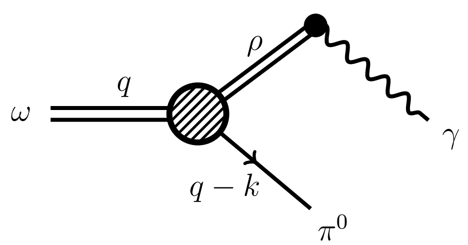
$$\begin{aligned}
i\mathcal{M}_{\text{VJP}} &= i\mathcal{M}^{(1)}_{\text{VJP}} + i\mathcal{M}^{(2)}_{\text{VJP}} + i\mathcal{M}^{(3)}_{\text{VJP}} + i\mathcal{M}^{(5)}_{\text{VJP}} + i\mathcal{M}^{(6)}_{\text{VJP}} \\
&= \left(-\frac{4\sqrt{2}ieC_1}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_r q^\rho k^\sigma(M^2_\omega - M^2_\pi)\right) + \left(\frac{2\sqrt{2}ieC_2}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_r q^\rho k^\sigma\left(M^2_\omega + M^2_\pi\right)\right) \\
&\quad + \left(-\frac{2\sqrt{2}ieC_3}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_r q^\rho k^\sigma\cdot 8M^2_\pi\right) + \left(\frac{2\sqrt{2}ieC_5}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_r q^\rho k^\sigma(M^2_\omega - M^2_\pi)\right) \\
&\quad + \left(-\frac{4\sqrt{2}ieC_6}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_r q^\rho k^\sigma M^2_\omega\right) \\
&= \frac{2\sqrt{2}ie}{M_V F M_\omega}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_r q^\rho k^\sigma\left[(C_2 - C_1 + C_5 - 2C_6)M^2_\omega + (C_1 + C_2 + 8C_3 - C_5)M^2_\pi\right]
\end{aligned}$$

11.3 VVP: vector, vector, pseudoscalar

VVP terms, vertices involving two vectors resonances and one pseudoscalar. Herein, ω and ρ^0 are vectors resonance, π is pseudoscalar.

$$\mathcal{O}^1_{\text{VVP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\alpha}\}\,\nabla_\alpha u^\sigma\rangle \qquad \mathcal{O}^2_{\text{VVP}} = i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\sigma}\}\,\chi_-\rangle$$

$$\mathcal{O}^3_{\text{VVP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, V^{\rho\alpha}\}\,u^\sigma\rangle \qquad \mathcal{O}^4_{\text{VVP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, V^{\rho\alpha}\}\,u_\alpha\rangle$$



First, we discuss ω to ρ^0 and π^0, then from ρ^0 to γ. VVP refers to the former one $\mathcal{L}_1$, and the latter one refers to:

$$\mathcal{L}_2 \supset \frac{F_v}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle \approx \frac{-eF_v}{\sqrt{2}}\langle V_{\mu\nu} Q(\partial^\mu \mathscr{A}^\nu - \partial^\nu \mathscr{A}^\mu)\rangle = \frac{-eF_v}{2}(\rho^0)_{\theta\varphi}(\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\beta)$$

$$_{\text{out}}\langle \gamma\pi^0|\omega\rangle_{\text{in}} = _{\text{out}}\langle\gamma|\rho^0\rangle\langle\rho^0\pi^0|\omega\rangle_{\text{in}} = \langle\gamma|iT_2|\rho^0\rangle\langle\rho^0\pi^0|iT_1|\omega\rangle = \langle\gamma\pi^0|iT_1 iT_2|\omega\rangle$$

Therefore, it's needed to calculate two Lagrangians respectively, and then the scattering amplitude.

11.3.1 $\mathcal{O}^1_{\text{VVP}}$

$$\begin{aligned}\mathcal{L} \supset d_1\mathcal{O}^1_{\text{VVP}} &= d_1\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\alpha}\}\nabla_\alpha u^\sigma\rangle \\ &\approx \frac{\sqrt{2}d_1}{F}\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\alpha}\}\partial_\alpha\partial^\sigma\phi\rangle \\ &= \frac{-2d_1}{F}\epsilon_{\mu\nu\rho\sigma}\left(\omega^{\mu\nu}(\rho^0)^{\rho\alpha} + (\rho^0)^{\mu\nu}\omega^{\rho\alpha}\right)\partial_\alpha\partial^\sigma\pi^0\end{aligned}$$

$$\begin{aligned}i\mathcal{M}_1 &= \frac{\delta^3}{\delta\mathscr{A}\delta\pi^0\delta\omega}\langle\gamma\pi^0|i\mathcal{L}_1 i\mathcal{L}_2|\omega\rangle \\
&\approx \frac{\delta^3}{\delta\mathscr{A}\delta\pi^0\delta\omega}\left\langle\gamma\pi^0\left|\frac{-2id_1}{F}\epsilon_{\mu\nu\rho\sigma}\left(\omega^{\mu\nu}(\rho^0)^{\rho\alpha} + (\rho^0)^{\mu\nu}\omega^{\rho\alpha}\right)\partial_\alpha\partial^\sigma\pi^0 \cdot \frac{-ieF_v}{2}(\rho^0)_{\theta\varphi}(\partial^\theta\mathscr{A}^\varphi - \partial^\varphi\mathscr{A}^\theta)\right|\omega\right\rangle \\
&= \frac{-ed_1F_v}{F}\epsilon_{\mu\nu\rho\sigma}\Bigg\{\frac{-i}{M_\omega}(q^\mu\epsilon^\nu_\omega - q^\nu\epsilon^\mu_\omega)\cdot i(q-k)_\alpha i(q-k)^\sigma\cdot i(k^\theta\epsilon^\varphi_\gamma - k^\varphi\epsilon^\theta_\gamma)\cdot\frac{i}{M_v^2}\frac{1}{M_v^2-k^2-i\epsilon} \\
&\quad \left[g^\rho_\theta g^\alpha_\varphi(M_v^2-k^2) + g^\rho_\theta k^\alpha k_\varphi - g^\rho_\varphi k^\alpha k_\theta - g^\alpha_\theta g^\rho_\varphi(M_v^2-k^2) - g^\alpha_\theta k^\rho k_\varphi + g^\alpha_\varphi k^\rho k_\theta\right] \\
&\quad + \frac{-i}{M_\omega}(q^\rho\epsilon^\alpha_\omega - q^\alpha\epsilon^\rho_\omega)\cdot i(q-k)_\alpha i(q-k)^\sigma\cdot i(k^\theta\epsilon^\varphi_\gamma - k^\varphi\epsilon^\theta_\gamma)\cdot\frac{i}{M_v^2}\frac{1}{M_v^2-k^2-i\epsilon} \\
&\quad \left[g^\mu_\theta g^\nu_\varphi(M_v^2-k^2) + g^\mu_\theta k^\nu k_\varphi - g^\mu_\varphi k^\nu k_\theta - g^\nu_\theta g^\mu_\varphi(M_v^2-k^2) - g^\nu_\theta k^\mu k_\varphi + g^\nu_\varphi k^\mu k_\theta\right]\Bigg\} \\
&= \frac{ied_1F_v}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\frac{1}{M_v^2-k^2-i\epsilon}\Bigg\{(q^\mu\epsilon^\nu_\omega - q^\nu\epsilon^\mu_\omega)(q-k)_\alpha(q-k)^\sigma\left[(k^\rho\epsilon^\alpha_\gamma - k^\alpha\epsilon^\rho_\gamma)M_v^2 - (k^\alpha\epsilon^\rho_\gamma\right. \\
&\quad \left. -k^\rho\epsilon^\alpha_\gamma)M_v^2\right] + (q^\rho\epsilon^\alpha_\omega - q^\alpha\epsilon^\rho_\omega)(q-k)_\alpha(q-k)^\sigma\left[(k^\mu\epsilon^\nu_\gamma - k^\nu\epsilon^\mu_\gamma)M_v^2 - (k^\nu\epsilon^\mu_\gamma - k^\mu\epsilon^\nu_\gamma)M_v^2\right]\Bigg\} \\
&= \frac{ied_1F_v}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\left[(q^\mu\epsilon^\nu_\omega - q^\nu\epsilon^\mu_\omega)(q-k)_\alpha(q-k)^\sigma(k^\rho\epsilon^\alpha_\gamma - k^\alpha\epsilon^\rho_\gamma)\right. \\
&\quad \left.+(q^\rho\epsilon^\alpha_\omega - q^\alpha\epsilon^\rho_\omega)(q-k)_\alpha(q-k)^\sigma(k^\mu\epsilon^\nu_\gamma - k^\nu\epsilon^\mu_\gamma)\right] \\
&= \frac{ied_1F_v}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\left[4k^\alpha(q-k)_\alpha q^\mu k^\sigma\epsilon^\nu_\omega\epsilon^\rho_\gamma + 4q^\alpha(k-q)_\alpha q^\sigma k^\mu\epsilon^\rho_\omega\epsilon^\nu_\gamma\right] \\
&= -\frac{ied_1F_v}{FM_\omega M_v^2}\epsilon_{\alpha\beta\rho\sigma}\epsilon^\alpha_\omega\epsilon^\beta_\gamma q^\rho k^\sigma M_\pi^2\end{aligned}$$

Below the calculation only with VVP,

$$
\begin{aligned}
i\mathcal{M}^1_{\text{VVP}} &= \frac{\delta^3}{\delta\rho^0\delta\pi^0\delta\omega}\langle\rho^0\pi^0|i\mathcal{L}^1_{\text{VVP}}|\omega\rangle \\
&\approx \frac{\delta^3}{\delta\rho^0\delta\pi^0\delta\omega}\left\langle\rho^0\pi^0\left|\frac{-2id_1}{F}\epsilon_{\mu\nu\rho\sigma}\left(\omega^{\mu\nu}(\rho^0)^{\rho\alpha}+(\rho^0)^{\mu\nu}\omega^{\rho\alpha}\right)\partial_\alpha\partial^\sigma\pi^0\right|\omega\right\rangle \\
&= -\frac{2id_1}{F}\epsilon_{\mu\nu\rho\sigma}\left[\frac{-i}{M_\omega}(q^\mu\epsilon^\nu_\omega-q^\nu\epsilon^\mu_\omega)\cdot\frac{i}{M_v}(k^\rho\epsilon^\alpha_\gamma-k^\alpha\epsilon^\rho_\gamma)\cdot\frac{1}{M_v}i(q-k)_\alpha i(q-k)^\sigma\right. \\
&\quad\left.+\frac{-i}{M_\omega}(q^\rho\epsilon^\alpha_\omega-q^\alpha\epsilon^\rho_\omega)\cdot\frac{i}{M_v}(k^\mu\epsilon^\nu_\gamma-k^\nu\epsilon^\mu_\gamma)\cdot\frac{1}{M_v}i(q-k)_\alpha i(q-k)^\sigma\right] \\
&= \frac{4id_1}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\cdot(q-k)_\alpha(q-k)^\sigma\cdot\left[q^\mu\epsilon^\nu_\omega\cdot(k^\rho\epsilon^\alpha_\gamma-k^\alpha\epsilon^\rho_\gamma)+k^\mu\epsilon^\nu_\gamma\cdot(q^\rho\epsilon^\alpha_\omega-q^\alpha\epsilon^\rho_\omega)\right] \\
&= \frac{4id_1}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\left[q^\mu\epsilon^\nu_\omega(-k^\alpha\epsilon^\rho_\gamma)(q-k)_\alpha(-k^\sigma)+k^\mu\epsilon^\nu_\gamma(-q^\alpha\epsilon^\rho_\omega)(q-k)_\alpha q^\sigma\right] \\
&= \frac{4id_1}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}q^\mu\epsilon^\nu k^\sigma\epsilon^\rho_\gamma(k-q)^\alpha(q-k)_\alpha = -\frac{4id_1}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_\gamma q^\rho k^\sigma M_\pi^2
\end{aligned}
$$

Two methods are different from eF_v. So we try only VVP terms next.

11.3.2 $\mathcal{O}^2_{\text{VVP}}$

$$
\begin{aligned}
\mathcal{L}\supset d_2\mathcal{O}^2_{\text{VVP}} &= id_2\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu},V^{\rho\sigma}\}\,\chi_-\rangle \\
&\approx id_2\epsilon_{\mu\nu\rho\sigma}\left\langle\{V^{\mu\nu},V^{\rho\sigma}\}\,(-\frac{\sqrt{2}i}{F})(\phi\chi+\chi\phi)\right\rangle \\
&= \frac{\sqrt{2}d_2}{F}\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu},V^{\rho\sigma}\}\,(\phi\chi+\chi\phi)\rangle \\
\#\text{coefficient} &= \frac{4M_\pi^2d_2}{F}\epsilon_{\mu\nu\rho\sigma}\left[\omega^{\mu\nu}(\rho^0)^{\rho\sigma}+(\rho^0)^{\mu\nu}\omega^{\rho\sigma}\right]\pi^0 \\
&= \frac{8M_\pi^2d_2}{F}\epsilon_{\mu\nu\rho\sigma}\omega^{\mu\nu}(\rho^0)^{\rho\sigma}\pi^0
\end{aligned}
$$

$$
\begin{aligned}
i\mathcal{M}^2_{\text{VVP}} &= \frac{\delta^3}{\delta\rho^0\delta\pi^0\delta\omega}\langle\rho^0\pi^0|i\mathcal{L}^2_{\text{VVP}}|\omega\rangle \\
&\approx \frac{\delta^3}{\delta\rho^0\delta\pi^0\delta\omega}\left\langle\rho^0\pi^0\left|\frac{8M_\pi^2d_2}{F}\epsilon_{\mu\nu\rho\sigma}\omega^{\mu\nu}(\rho^0)^{\rho\sigma}\pi^0\right|\omega\right\rangle \\
&= \frac{8iM_\pi^2d_2}{F}\epsilon_{\mu\nu\rho\sigma}\cdot\frac{-i}{M_\omega}(q^\mu\epsilon^\nu_\omega-q^\nu\epsilon^\mu_\omega)\cdot\frac{i}{M_v}(k^\rho\epsilon^\sigma_\gamma-k^\sigma\epsilon^\rho_\gamma)\cdot\frac{1}{M_v} \\
&= \frac{8iM_\pi^2d_2}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\cdot 2q^\mu\epsilon^\nu_\omega\cdot 2k^\rho\epsilon^\sigma_\gamma \\
&= -\frac{32iM_\pi^2d_2}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_\gamma q^\rho k^\sigma
\end{aligned}
$$

11.3.3 $\mathcal{O}^3_{\mathrm{VVP}}$

$$\begin{aligned}\mathcal{L} \supset d_3\mathcal{O}^3_{\mathrm{VVP}} &= d_3\epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, V^{\rho\alpha}\}\, u^\sigma\rangle \\ &\approx d_3\epsilon_{\mu\nu\rho\sigma}\left\langle\{\nabla_\alpha V^{\mu\nu}, V^{\rho\alpha}\}\,(-\frac{\sqrt{2}}{F})\partial^\sigma\phi\right\rangle \\ \# &= -\frac{\sqrt{2}d_3}{F}\epsilon_{\mu\nu\rho\sigma}\sqrt{2}\left[\partial_\alpha\omega^{\mu\nu}(\rho^0)^{\rho\alpha} + \partial_\alpha(\rho^0)^{\mu\nu}\omega^{\rho\alpha}\right]\partial^\sigma\pi^0 \\ &= -\frac{2d_3}{F}\epsilon_{\mu\nu\rho\sigma}\left[\partial_\alpha\omega^{\mu\nu}(\rho^0)^{\rho\alpha} + \partial_\alpha(\rho^0)^{\mu\nu}\omega^{\rho\alpha}\right]\partial^\sigma\pi^0\end{aligned}$$

$$\begin{aligned}i\mathcal{M}^3_{\mathrm{VVP}} &= \frac{\delta^3}{\delta\rho^0\delta\pi^0\delta\omega}\langle\rho^0\pi^0|i\mathcal{L}^3_{\mathrm{VVP}}|\omega\rangle \\ &\approx \frac{\delta^3}{\delta\rho^0\delta\pi^0\delta\omega}\left\langle\rho^0\pi^0\left|-\frac{2id_3}{F}\epsilon_{\mu\nu\rho\sigma}\left[\partial_\alpha\omega^{\mu\nu}(\rho^0)^{\rho\alpha} + \partial_\alpha(\rho^0)^{\mu\nu}\omega^{\rho\alpha}\right]\partial^\sigma\pi^0\right.\right\rangle \\ &= -\frac{2id_3}{F}\epsilon_{\mu\nu\rho\sigma}\left[(-i)q_\alpha\cdot\frac{-i}{M_\omega}(q^\mu\epsilon^\nu_\omega - q^\nu\epsilon^\mu_\omega)\cdot\frac{i}{M_v}(k^\rho\epsilon^\alpha_\gamma - k^\alpha\epsilon^\rho_\gamma)\cdot\frac{i}{M_v}(q-k)^\sigma\right. \\ &\quad\left. +(+i)k_\alpha\cdot\frac{i}{M_v}(k^\mu\epsilon^\nu_\gamma - k^\nu\epsilon^\mu_\gamma)\cdot\frac{-i}{M_\omega}(q^\rho\epsilon^\alpha_\omega - q^\alpha\epsilon^\rho_\omega)\cdot\frac{i}{M_v}(q-k)^\sigma\right] \\ &= -\frac{2id_3}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\left[q_\alpha(q^\mu\epsilon^\nu_\omega - q^\nu\epsilon^\mu_\omega)\cdot(-k^\alpha\epsilon^\rho_\gamma)(-k)^\sigma - k_\alpha(k^\mu\epsilon^\nu_\gamma - k^\nu\epsilon^\mu_\gamma)(-q^\alpha\epsilon^\rho_\omega)q^\sigma\right] \\ &= -\frac{4id_3}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\left(q^\mu\epsilon^\nu_\omega k^\sigma\epsilon^\rho_\gamma\cdot q_\alpha k^\alpha + k^\mu\epsilon^\nu_\gamma q^\sigma\epsilon^\rho_\omega\cdot q_\alpha k^\alpha\right) \\ &= -\frac{4id_3}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_\gamma q^\rho k^\sigma\cdot(M^2_\omega - M^2_\pi)\end{aligned}$$

11.3.4 $\mathcal{O}^4_{\mathrm{VVP}}$

$$\begin{aligned}\mathcal{L} \supset d_4\mathcal{O}^4_{\mathrm{VVP}} &= d_4\epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, V^{\rho\alpha}\}\, u_\alpha\rangle \\ &\approx d_4\epsilon_{\mu\nu\rho\sigma}\left\langle\{\partial^\sigma V^{\mu\nu}, V^{\rho\alpha}\}\left(-\frac{\sqrt{2}}{F}\partial_\alpha\phi\right)\right\rangle = 0\end{aligned}$$

Because $\epsilon_{\mu\nu\rho\sigma}\partial^\sigma V^{\mu\nu} \propto \epsilon_{\mu\nu\rho\sigma}\left[k^\sigma k^\mu\epsilon^\nu_\omega - k^\sigma k^\nu\epsilon^\mu_\omega\right] = 0$, the product of two momenta with indices inside Levi-Civita symbol equals to 0.

11.3.5 $\mathcal{L}_{\mathrm{VVP}}$

Total Lagrangian of VVP terms is the sum of each term,

$$\mathcal{L}_{\mathrm{VVP}} = \sum_{a=1}^{4} d_a\mathcal{O}^a_{\mathrm{VVP}}$$

namely the total $\mathcal{L}_1$. Multiply $\mathcal{L}_2$ and we get the final scattering amplitude.

$$
\begin{aligned}
i\mathcal{M}_{\mathrm{VVP}} &= \frac{\delta^3}{\delta\rho^0\delta\pi^0\delta\omega}\langle\rho^0\pi^0|i\mathcal{L}_{\mathrm{VVP}}|\omega\rangle \\
&= i\mathcal{M}^1_{\mathrm{VVP}} + i\mathcal{M}^2_{\mathrm{VVP}} + i\mathcal{M}^3_{\mathrm{VVP}} + i\mathcal{M}^4_{\mathrm{VVP}} \\
&= \left(-\frac{4ied_1F_v}{FM_\omega M_v^2}\epsilon_{\alpha\beta\rho\sigma}\epsilon^\alpha_\omega\epsilon^\beta_\gamma q^\rho k^\sigma \cdot M_\pi^2\right) + \left(-\frac{32ieF_vd_2}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_\gamma q^\rho k^\sigma \cdot M_\pi^2\right) \\
&\quad + \left(-\frac{4ieF_vd_3}{FM_\omega M_v^2}\epsilon_{\mu\nu\rho\sigma}\epsilon^\mu_\omega\epsilon^\nu_\gamma q^\rho k^\sigma \cdot (M_\omega^2 - M_\pi^2)\right) + 0 \\
&= -\frac{4ied_1F_v}{FM_\omega M_v^2}\epsilon_{\alpha\beta\rho\sigma}\epsilon^\alpha_\omega\epsilon^\beta_\gamma q^\rho k^\sigma\left[(d_1 + 8d_2 - d_3)M_\pi^2 + d_3M_\omega^2\right]
\end{aligned}
$$

11.3.6 Discussion

Herein, we validate that $|\rho^0\rangle\langle\rho^0|$ would introduce eF_v.

$$
{}_{\mathrm{out}}\langle\gamma\pi^0|\omega\rangle_{\mathrm{in}} = {}_{\mathrm{out}}\langle\gamma|\rho^0\rangle\langle\rho^0\pi^0|\omega\rangle_{\mathrm{in}} = \langle\gamma|iT_2|\rho^0\rangle\langle\rho^0\pi^0|iT_1|\omega\rangle = \langle\gamma\pi^0|iT_1iT_2|\omega\rangle
$$

$$
\mathcal{L}_2 \supset \frac{F_v}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle \approx \frac{-eF_v}{\sqrt{2}}\langle V_{\mu\nu}Q(\partial^\mu\mathscr{A}^\nu - \partial^\nu\mathscr{A}^\mu)\rangle = \frac{-eF_v}{2}(\rho^0)_{\theta\varphi}(\partial^\theta\mathscr{A}^\varphi - \partial^\varphi\mathscr{A}^\beta)
$$

$[(\rho^0)^{\rho\alpha} + (\rho^0)^{\mu\nu}]$ with $\langle\rho^0|$, as well as $(\rho^0)_{\theta\varphi}$ with $|\rho^0\rangle$. Take $(\rho^0)^{\rho\alpha}(\rho^0)_{\theta\varphi}$ as an instance, (the same procedure for $(\rho^0)^{\mu\nu}(\rho^0)_{\theta\varphi}$, by changing indices.)

$$
\begin{aligned}
&\cdots\langle\rho^0|\,(\rho^0)^{\rho\alpha}(\rho^0)_{\theta\varphi}\,|\rho^0\rangle \\
&= i(k^\rho\epsilon^\alpha - k^\alpha\epsilon^\rho)\cdot(-i)(k_\theta\epsilon_\varphi - k_\varphi\epsilon_\theta) \\
&= (k^\rho\epsilon^\alpha k_\theta\epsilon_\varphi - k^\rho\epsilon^\alpha k_\varphi\epsilon_\theta - k^\alpha\epsilon^\rho k_\theta\epsilon_\varphi + k^\alpha\epsilon^\rho k_\varphi\epsilon_\theta) \\
&= (k^\rho\epsilon^\alpha - k^\alpha\epsilon^\rho)k_\theta\epsilon_\varphi - (k^\rho\epsilon^\alpha - k^\alpha\epsilon^\rho)k_\varphi\epsilon_\theta \\
&\sim (k^\rho\epsilon^\alpha - k^\alpha\epsilon^\rho)g^\varphi_\theta k_\varphi k_\varphi - (k^\rho\epsilon^\alpha - k^\alpha\epsilon^\rho)g^\theta_\varphi k_\theta k_\theta \\
&= (k^\rho\epsilon^\alpha - k^\alpha\epsilon^\rho)M_v^2 - (k^\rho\epsilon^\alpha - k^\alpha\epsilon^\rho)k^2 \\
&= 2k^\rho\epsilon^\alpha(M_v^2 - k^2)
\end{aligned}
$$

$(M_v^2 - k^2)$ in the final result vanishes due to $\frac{1}{M_v^2 - k^2 - i\epsilon}$, and the latter was introduced by Feynman propagator, with the minus symbol from $\sum\epsilon^*_\mu\epsilon_\nu \to g_{\mu\nu}$. Moreover, coefficient 2 vanishes by $\frac{-eF_v}{2}$, leaving eF_v only. Note that $\sim$ refers to ϵ_θ replaced by k_θ, like that in Ward identity. With on-shell relation $m^2 = k^2$, we write $M_v^2 - k^2$.

Certainly, the result would always be the same as tensor propagators.

$\mu\nu$ ═══════ $\rho\sigma$

$$
\begin{aligned}
&\langle 0\,|T\,\{W_{\mu\nu}(x), W_{\rho\sigma}(y)\}|\,0\rangle \\
=&\frac{i}{M^2}\int\frac{d^4ke^{-ik(x-y)}}{(2\pi)^4(M^2 - k^2 + i\epsilon)}\left[g_{\mu\rho}g_{\nu\sigma}(M^2 - k^2) + g_{\mu\rho}k_\nu k_\sigma - g_{\mu\sigma}k_\nu k_\rho - (\mu\leftrightarrow\nu)\right]
\end{aligned}
$$

12 Summary

12.1 Process of calculation

Peskin:

1. Draw Feynman diagrams, designate momenta etc;
2. Determine initial and final states, write eq. 4.71 and 4.73;
3. Write $\mathcal{M}$ according to Feynman rules;
4. Calculate$\mathcal{M}$ by trace tech;

Primer:

1. Determine Lagrangian terms according to the symmetry;
2. Calculate VJP and VVP terms;
3. Obtain correlation function by functional derivatives;
4. Add all terms to get the total scattering amplitude;

12.2 Functionals and Local Functional Derivatives

Appendix B. of Primer.

Local functional derivatives are natural generalizaiton so f classical partial derivatives to infinite dimensions. Let $\mathscr{F}$ denote the set of all functions $j : \mathbb{R}^n \to \mathbb{K}(\mathbb{K} = \mathbb{R}$ or $\mathbb{C})$. A real (complex) functional is a map $j \mapsto Z[j]$ from $\mathscr{F}$ to $\mathbb{R}$ ($\mathbb{C}$), which assigns a real (complex) number to $Z[j]$ to each function j. A typical example is given by an integral of the type

$$F[j] = \int d^n x g[j(x)]$$

with g an integrable function. Let j be a function of two sets of variables, collectively denoted by x and y; $F[j(y)]$ denotes a functional which depends on the values of j for all x at fixed y.

Consider a definition of partial functional derivatives based on the Dirac delta function,

$$\delta_y : \begin{cases} \mathbb{R}^n \to \mathbb{R} \\ x \mapsto \delta_y(x) = \delta^n(x-y) \end{cases}$$

In terms of the Dirac delta function the partial functional derivative is defined as

$$\frac{\delta F[f]}{\delta f(y)} \equiv \lim_{\epsilon \to 0} \frac{F[f + \epsilon \delta_y] - F[f]}{\epsilon} \tag{B.1}$$

Partial functional derivatives share basic properties with ordinary partial derivatives, namely linearity & product rule & chain rule.

Define $f(y)$ as the functional

$$f(y) = F_y[f] = \int d^n z \delta^n(y-z) f(z)$$

and then apply B.1. An important rule for the local functional derivative of a function is

$$\frac{\delta f(y)}{\delta f(x)} = \delta^n(y-x) \tag{B.2}$$

Analogously we have

$$\frac{\delta g[f(y)]}{\delta f(x)} = \delta^n(y-x) g'[f(y)]$$

$$\frac{\delta^k g[f(y)]}{\delta f(x_k) \cdots \delta f(x_1)} = \delta^n(y-x_k) \cdots \delta^n(y-x_1) g^{(k)}[f(y)]$$

Consider the generating functional of Green functions as partial functional derivatives

$$G_n(x_1, \cdots, x_n) = \langle 0 | T[\phi(x_1) \cdots \phi(x_n)] | 0 \rangle$$

of a real scalar field operator ϕ whose dynamics is determined by a Lagrangian $\mathscr{L}_{\text{ext}} = j(x)\phi(x)$. Note

$$f(x) = a_0 + a_1 x + \frac{1}{2} a_2 x^2 + \cdots \qquad \Rightarrow \qquad \frac{d^n f}{dx^n}(x=0) = a_n$$

the generating functional for the Green functions G_n is given by

$$\exp(iZ[j]) = \left\langle 0 \left| T \exp\left[i \int d^4 x \mathscr{L}_{\text{ext}}(x) \right] \right| 0 \right\rangle$$
$$= 1 + i \int d^4 x j(x) \langle 0 | \phi(x) | 0 \rangle + \sum_{k=2}^{\infty} \frac{i^k}{k!} \int d^4 x_1 \cdots d^4 x_k j(x_1) \cdots j(x_k) \langle 0 | T[\phi(x_1) \cdots \phi(x_k)] | 0 \rangle$$

Some remarks:

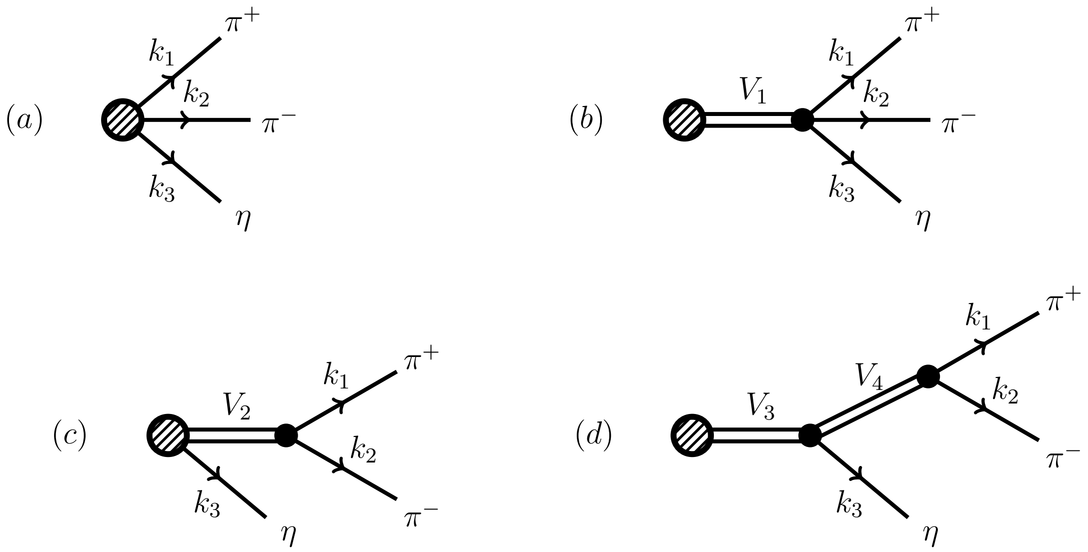
1. Sometimes $\exp(iZ[j])$ written as $Z[j]$, and $Z[j]$ as $W[j]$ ... ...
2. j represents a function and can thus be taken out of the matrix element.
3. Select the proper expansion of order for Green functions, i.e. second order for G_2

$$G_2(x_1, x_2) = (-i)^2 \frac{\delta^2 \exp(iZ[j])}{\delta j(x_1) j(x_2)} \bigg|_{j=0}$$

Part IV
ee to pi pi eta

Ref: Dai, Ling Yun, Jorge Portolés, and Olga Shekhovtsova. "Three pseudoscalar meson production in e+ e annihilation." *Physical Review D* 88.5 (2013): 056001.

13 Feynman diagrams



14 Notions & Parameters

14.1 Form factors

$\mathcal{M}^\eta$, the amplitude for the process, with hadronic matrix element T_μ^η

$$\mathcal{M}^\eta = -\frac{4\pi\alpha}{Q^2} T_\mu^\eta \bar{v}(k')\gamma^\mu u(k) = -\frac{4\pi\alpha}{Q^2} \cdot iF_V^\eta \cdot \epsilon_{\mu\nu\alpha\beta} k_1^\nu k_2^\alpha k_3^\beta \bar{v}(k')\gamma^\mu u(k)$$
$$T_\mu^\eta \equiv \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3) \,\middle|\, \mathcal{J}_\mu^{\text{em}} e^{i\mathcal{L}_{\text{QCD}}} \,\middle|\, 0 \right\rangle$$

In elementary particle physics and mathematical physics, a form factor is a function that encapsulates the properties of a certain particle interaction w/o including all of the underlying physics, but instead, providing the momentum dependence of suitable matrix elements.

$$\boxed{\left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3) \,\middle|\, \mathcal{J}_\mu^{\text{em}} e^{i\mathcal{L}_{\text{QCD}}} \,\middle|\, 0 \right\rangle = iF_V^\eta(Q^2, s, t) \cdot \epsilon_{\mu\nu\alpha\beta} k_1^\nu k_2^\alpha k_3^\beta}$$

Here the Mandelstam variables:

$$Q = k + k' \qquad E_{\text{CM}} \equiv \sqrt{Q^2} \qquad s = (Q - k_3)^2 \qquad t = (Q - k_1)^2 \qquad u = Q^2 + 2m_\pi^2 + m_\eta^2 - s - t$$

The final result is furnished by the sum of four diagram contributions as:

$$F_V^\eta(Q^2, s, t) = F_a^\eta + F_b^\eta + F_c^\eta + F_d^\eta$$

The amplitude for three-meson production in e^-e^+ collisions at low energies is dominantly driven by the electromagnetic current below:

$$\begin{aligned}
\mathcal{J}_\mu^{\text{em}} &= \mathcal{V}_\mu^3 + \mathcal{V}_\mu^8/\sqrt{3} \qquad\qquad \left(\mathcal{V}_\mu^i = \bar{q}\gamma_\mu \frac{\lambda^i}{2} q\right) \\
&= \bar{q}\gamma_\mu \frac{\lambda^3}{2} q + \bar{q}\gamma_\mu \frac{\lambda^8}{2} q/\sqrt{3} \\
&= \frac{1}{2}\bar{q}\gamma_\mu \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \right] q = \frac{1}{3}\bar{q}\gamma_\mu \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} q
\end{aligned}$$

14.2 Lagrangians

$$\begin{aligned}
\mathcal{L}_{\text{R}\chi\text{T}} =& \mathcal{L}_{(2)}^{\text{GB}} + \mathcal{L}_{(4)}^{\text{GB}} + \mathcal{L}_{\text{V}} \\
& \mathcal{L}_{(2)}^{\text{GB}} = \mathcal{L}_{(2)}^{\chi\text{PT}} = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+\rangle \\
& \mathcal{L}_{(4)}^{\text{GB}} = \frac{iN_C\sqrt{2}}{12\pi^2F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle \partial^\mu\Phi\partial^\nu\Phi\partial^\rho\Phi v^\sigma\right\rangle + \cdots \\
& \mathcal{L}_{\text{V}} = \mathcal{L}_{\text{kin}}^{\text{V}} + \mathcal{L}_{\text{int}}^{\text{V}} = \left(-\frac{1}{2}\left\langle \nabla^\lambda V_{\lambda\mu}\nabla_\nu V^{\nu\mu}\right\rangle + \frac{1}{4}M_V^2\left\langle V_{\mu\nu}V^{\mu\nu}\right\rangle\right) + \boxed{\mathcal{L}_{(2)}^{\text{V}} + \mathcal{L}_{(4)}^{\text{V}} + \mathcal{L}_{(2)}^{\text{VV}}}
\end{aligned}$$

$$\begin{cases}
\mathcal{L}_{(2)}^{\text{V}} = \dfrac{F_V}{2\sqrt{2}}\left\langle V_{\mu\nu}f_+^{\mu\nu}\right\rangle + \dfrac{iG_V}{\sqrt{2}}\left\langle V_{\mu\nu}u^\mu u^\nu\right\rangle \\
\mathcal{L}_{(4)}^{\text{V}} = \displaystyle\sum_{j=1}^{7}\frac{c_j}{M_V}\mathcal{O}_{\text{VJP}}^j + \sum_{j=1}^{5}\frac{g_j}{M_V}\mathcal{O}_{\text{VPPP}}^j \\
\mathcal{L}_{(2)}^{\text{VV}} = \displaystyle\sum_{j=1}^{4} d_j\mathcal{O}_{\text{VVP}}^j
\end{cases}$$

$$\Phi = \begin{pmatrix} \eta/\sqrt{6} & \pi^+ & . \\ \pi^- & \eta/\sqrt{6} & . \\ . & . & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

14.3 Mixing angle

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P \end{pmatrix}\begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix} \qquad\qquad \begin{pmatrix} V_\mu^8 \\ V_\mu^0 \end{pmatrix} = \begin{pmatrix} \cos\theta_V & -\sin\theta_V \\ \sin\theta_V & \cos\theta_V \end{pmatrix}\begin{pmatrix} \phi_8 \\ \omega_0 \end{pmatrix}$$

Suppose

$$\sin\theta_P = 0 \qquad\qquad \sin\theta_V = 1/\sqrt{3}$$

15 Calculations w/o mixing angles

15.1 $\mathcal{L}_{(4)}^{\mathrm{GB}}$ term

$$\mathcal{L}_{(4)}^{\mathrm{GB}} = \frac{iN_C\sqrt{2}}{12\pi^2F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\partial^\mu\Phi\partial^\nu\Phi\partial^\rho\Phi v^\sigma\right\rangle$$

$$= \frac{iN_C\sqrt{2}}{12\pi^2F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\frac{1}{\sqrt{6}}\partial^{\mu\nu\rho}\begin{pmatrix} \pi^+\pi^-\eta & & \\ \eta\pi^+\pi^- & . & . \\ \pi^+\eta\pi^- & & \\ & \pi^-\eta\pi^+ & \\ . & \eta\pi^-\pi^+ & . \\ & \pi^-\pi^+\eta & \\ . & . & 0\end{pmatrix}(-eQ\mathscr{A}^\sigma)\right\rangle$$

$$= \frac{iN_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\partial^{\mu\nu\rho}\begin{pmatrix} \pi^+\pi^-\eta & & \\ \eta\pi^+\pi^- & . & . \\ \pi^+\eta\pi^- & & \\ & \pi^-\eta\pi^+ & \\ . & \eta\pi^-\pi^+ & . \\ & \pi^-\pi^+\eta & \\ . & . & 0\end{pmatrix}\cdot\left(\frac{-e\mathscr{A}^\sigma}{3}\right)\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}\right\rangle$$

$$= \frac{iN_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\mu\nu\rho\sigma}\cdot\partial^{\mu\nu\rho}\left[2\begin{pmatrix}\pi^+\pi^-\eta \\ \eta\pi^+\pi^- \\ \pi^+\eta\pi^-\end{pmatrix} - \begin{pmatrix}\pi^-\eta\pi^+ \\ \eta\pi^-\pi^+ \\ \pi^-\pi^+\eta\end{pmatrix}\right]\left(\frac{-e\mathscr{A}^\sigma}{3}\right)$$

$$= \frac{iN_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\mu\nu\rho\sigma}\partial^\mu\pi^+\partial^\nu\pi^-\partial^\rho\eta\cdot(-e\mathscr{A}^\sigma)$$

$$T_\mu^\eta \equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\mathrm{em}}e^{i\mathcal{L}_{\mathrm{QCD}}}\right|0\right\rangle$$

$$\approx \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\mathrm{em}}\cdot(1+i\mathcal{L}_{\mathrm{QCD}})\right|0\right\rangle$$

$$= \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\mathrm{em}}\cdot i\mathcal{L}_{\mathrm{QCD}}\right|0\right\rangle$$

$$= \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\mathrm{em}}\cdot\frac{N_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\mu\nu\rho\sigma}\partial^\mu\pi^+\partial^\nu\pi^-\partial^\rho\eta\cdot(e\mathscr{A}^\sigma)\right|0\right\rangle$$

$$= \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\mathrm{em}}\cdot\frac{N_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\sigma\nu\rho\mu}\partial^\sigma\pi^+\partial^\nu\pi^-\partial^\rho\eta\cdot(e\mathscr{A}^\mu)\right|0\right\rangle$$

$$= \frac{-N_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\sigma\nu\rho\mu}\cdot ik_1^\sigma\cdot ik_2^\nu\cdot ik_3^\rho = \frac{iN_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\sigma\nu\rho\mu}k_1^\sigma k_2^\nu k_3^\rho$$

$$= \frac{iN_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\sigma\nu\rho\mu}k_1^\nu k_2^\rho k_3^\sigma = \frac{-iN_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma$$

$$= \frac{-iN_C}{12\sqrt{3}\pi^2F^3}\epsilon_{\mu\nu\alpha\beta}k_1^\nu k_2^\alpha k_3^\beta = iF_a^\eta(Q^2,s,t)\cdot\epsilon_{\mu\nu\alpha\beta}k_1^\nu k_2^\alpha k_3^\beta$$

$$F_a^\eta = -\frac{N_C}{12\sqrt{3}\pi^2F^3}$$

15.2 VPPP terms

$$\mathcal{L}^{\mathrm{V}}_{(4)} = \sum_{j=1}^{7} \frac{c_j}{M_V} \mathcal{O}^j_{\mathrm{VJP}} + \sum_{j=1}^{5} \frac{g_j}{M_V} \mathcal{O}^j_{\mathrm{VPPP}}$$

$$\mathcal{O}^1_{\mathrm{VPPP}} = i\epsilon_{\mu\nu\alpha\beta} \left\langle V^{\mu\nu}(h^{\alpha\gamma}u_\gamma u^\beta - u^\beta u_\gamma h^{\alpha\gamma})\right\rangle \qquad \mathcal{O}^2_{\mathrm{VPPP}} = i\epsilon_{\mu\nu\alpha\beta} \left\langle V^{\mu\nu}(h^{\alpha\gamma}u^\beta u_\gamma - u_\gamma u^\beta h^{\alpha\gamma})\right\rangle$$

$$\mathcal{O}^3_{\mathrm{VPPP}} = i\epsilon_{\mu\nu\alpha\beta} \left\langle V^{\mu\nu}(u_\gamma h^{\alpha\gamma}u^\beta - u^\beta h^{\alpha\gamma} u_\gamma)\right\rangle \qquad \mathcal{O}^4_{\mathrm{VPPP}} = \epsilon_{\mu\nu\alpha\beta} \left\langle \left\{V^{\mu\nu}, u^\alpha u^\beta\right\} \chi_-\right\rangle$$

$$\mathcal{O}^5_{\mathrm{VPPP}} = \epsilon_{\mu\nu\alpha\beta} \left\langle u^\alpha V^{\mu\nu} u^\beta \chi_-\right\rangle$$

15.2.1 $\mathcal{O}^1_{\mathrm{VPPP}}$

$$u_\mu = i\left[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger\right] \approx -\frac{\sqrt{2}\partial_\mu\phi}{F}$$

$$h_{\mu\nu} = \nabla_\mu u_\nu + \nabla_\nu u_\mu \approx -\frac{\sqrt{2}}{F}(\partial_\nu\partial_\mu + \partial_\mu\partial_\nu)\phi$$

$$f_+^{\mu\nu} = uF_L^{\mu\nu}u^\dagger + u^\dagger F_R^{\mu\nu}u \approx -2eQ(\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu)$$

$$\begin{aligned}
\mathcal{L}^1_{\mathrm{VPPP}} &= \frac{g_1}{M_V}\cdot i\epsilon_{\mu\nu\alpha\beta}\left\langle V^{\mu\nu}(h^{\alpha\gamma}u_\gamma u^\beta - u^\beta u_\gamma h^{\alpha\gamma})\right\rangle \\
&= \frac{g_1}{M_V}\cdot i\epsilon_{\mu\nu\alpha\beta}\left\langle V^{\mu\nu}\left[\left(-\frac{\sqrt{2}}{F}\right)(\partial^\alpha\partial^\gamma + \partial^\gamma\partial^\alpha)\phi\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial_\gamma\phi\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial^\beta\phi - \cdots\right]\right\rangle \\
&= \frac{-2\sqrt{2}ig_1}{\sqrt{3}F^3M_V}\epsilon_{\mu\nu\alpha\beta}\left[\rho^{\mu\nu}\left(\partial^\alpha\partial^\gamma + \partial^\gamma\partial^\alpha\right)\pi^+\partial_\gamma\pi^-\partial^\beta\eta - \rho^{\mu\nu}\partial^\beta\pi^+\partial_\gamma\pi^-\left(\partial^\alpha\partial^\gamma + \partial^\gamma\partial^\alpha\right)\eta\right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{\mathrm{V}}_{(2)} &= \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle + \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle \supset \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle = \frac{-eF_V}{2}\rho_{\mu\nu}(\partial^\mu\mathscr{A}^\nu - \partial^\nu\mathscr{A}^\mu) \\
&= \frac{F_V}{2\sqrt{2}}\left\langle\frac{1}{\sqrt{2}}\begin{pmatrix}\rho & . & . \\ . & -\rho & . \\ . & . & 0\end{pmatrix}_{\mu\nu}\cdot(-2e/3)\begin{pmatrix}2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}(\partial^\mu\mathscr{A}^\nu - \partial^\nu\mathscr{A}^\mu)\right\rangle
\end{aligned}$$

$$
\begin{aligned}
T^\eta_\mu &\equiv \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\rm em}_\mu e^{i\mathcal{L}_{\rm QCD}}\right|0\right\rangle \approx \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\rm em}_\mu\cdot i\mathcal{L}^1_{\rm VPPP}\cdot i\mathcal{L}^{\rm V}_{(2)}\right|0\right\rangle \\
&= \left\langle \pi^+\pi^-\eta\left|-\mathcal{J}^{\rm em}_\mu\cdot\frac{-4ig_1}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\alpha\beta}\rho^{\mu\nu}\left[\partial^{\alpha\gamma}\partial_\gamma\partial^\beta(\cdots)-\partial^\beta\partial_\gamma\partial^{\alpha\gamma}(\cdots)\right]\cdot\frac{-eF_V}{2}\rho_{\theta\varphi}(\partial^\theta\mathscr{A}^\varphi-\partial^\varphi\mathscr{A}^\theta)\right|0\right\rangle \\
&= \frac{2ieF_Vg_1\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_VF^3}\left\langle \pi^+\pi^-\eta\left|-\mathcal{J}^{\rm em}_\mu\cdot\left[\partial^{\alpha\gamma}\partial_\gamma\partial^\beta(\cdots)-\partial^\beta\partial_\gamma\partial^{\alpha\gamma}(\cdots)\right]\cdot\frac{i(g^\mu_\theta g^\nu_\varphi-g^\nu_\theta g^\mu_\varphi)}{M_\rho^2-Q^2}(\partial^\theta\mathscr{A}^\varphi-\partial^\varphi\mathscr{A}^\theta)\right|0\right\rangle \\
&= \frac{-4eF_Vg_1\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_VF^3(M_\rho^2-Q^2)}\left\langle \pi^+\pi^-\eta\left|-\mathcal{J}^{\rm em}_\mu\cdot\left[\partial^{\alpha\gamma}\partial_\gamma\partial^\beta(\cdots)-\partial^\beta\partial_\gamma\partial^{\alpha\gamma}(\cdots)\right]\cdot(-2)\partial^\nu\mathscr{A}^\mu\right|0\right\rangle \\
&= \frac{8F_Vg_1\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_VF^3(M_\rho^2-Q^2)}(-iQ^\nu)\left[\begin{pmatrix} 1^\alpha1^\gamma2_\gamma3^\beta & & 2^\alpha2^\gamma3_\gamma1^\beta \\ 3^\alpha3^\gamma1_\gamma2^\beta & - & 2^\alpha2^\gamma1_\gamma3^\beta \\ 1^\alpha1^\gamma3_\gamma2^\beta & & 3^\alpha3^\gamma2_\gamma1^\beta \end{pmatrix} - \begin{pmatrix} 1^\beta2_\gamma3^\alpha3^\gamma & & 2^\beta3_\gamma1^\alpha1^\gamma \\ 3^\beta1_\gamma2^\alpha2^\gamma & - & 2^\beta1_\gamma3^\alpha3^\gamma \\ 1^\beta3_\gamma2^\alpha2^\gamma & & 3^\beta2_\gamma1^\alpha1^\gamma \end{pmatrix}\right] \\
&= \frac{-4\sqrt{6}iF_Vg_1\epsilon_{\mu\nu\alpha\beta}}{3M_VF^3(M_\rho^2-Q^2)}1^\nu2^\alpha3^\beta(-12-13+13+23-12-23-23-12+23+13-13-12) \\
&= \left[\frac{8\sqrt{6}iF_Vg_1\epsilon_{\mu\nu\alpha\beta}}{3M_VF^3(M_\rho^2-Q^2)}k_1^\nu k_2^\alpha k_3^\beta\right]\cdot 2(k_1k_2) = [\cdots]\cdot(s-2m_\pi^2)
\end{aligned}
$$

$$
F^\eta_b(1) = \frac{8\sqrt{6}F_V}{3M_VF^3(M_\rho^2-Q^2)}\cdot g_1(s-2m_\pi^2)
$$

15.2.2 $\mathcal{O}^2_{\rm VPPP}$

Note, 1 refers to π^+, 2 refers to π, 3 refers to η; or k_1, k_2, k_3 if with indices.

$$
\begin{aligned}
\mathcal{L}^2_{\rm VPPP} &= \frac{g_2}{M_V}\mathcal{O}^2_{\rm VPPP} = \frac{g_2}{M_V}\cdot i\epsilon_{\mu\nu\alpha\beta}\left\langle V^{\mu\nu}(h^{\alpha\gamma}u^\beta u_\gamma-u_\gamma u^\beta h^{\alpha\gamma})\right\rangle \\
&= \frac{g_2}{M_V}\cdot i\epsilon_{\mu\nu\alpha\beta}\left\langle V^{\mu\nu}\left[\left(-\frac{\sqrt{2}}{F}\right)(\partial^\alpha\partial^\gamma+\partial^\gamma\partial^\alpha)\phi\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial^\beta\phi\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial_\gamma\phi-\cdots\right]\right\rangle \\
&= \frac{-2\sqrt{2}ig_2}{\sqrt{3}F^3M_V}\epsilon_{\mu\nu\alpha\beta}\left[\rho^{\mu\nu}\left(\partial^\alpha\partial^\gamma+\partial^\gamma\partial^\alpha\right)\pi^+\partial^\beta\pi^-\partial_\gamma\eta-\rho^{\mu\nu}\partial_\gamma\pi^+\partial^\beta\pi^-\left(\partial^\alpha\partial^\gamma+\partial^\gamma\partial^\alpha\right)\eta\right] \\
&= \frac{-4i}{\sqrt{6}F^3}\epsilon_{\mu\nu\alpha\beta}\rho^{\mu\nu}\left[\partial^{\alpha\gamma}\partial^\beta\partial_\gamma\begin{pmatrix}123 & & 231\\ 312 & - & 213\\ 132 & & 321\end{pmatrix}-\partial_\gamma\partial^\beta\partial^{\alpha\gamma}\begin{pmatrix}123 & & 231\\ 312 & - & 213\\ 132 & & 321\end{pmatrix}\right]
\end{aligned}
$$

$$T^\eta_\mu \equiv \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\text{em}}_\mu e^{i\mathcal{L}_{\text{QCD}}}\right|0\right\rangle \approx \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\text{em}}_\mu \cdot i\mathcal{L}^2_{\text{VPPP}} \cdot i\mathcal{L}^{\text{V}}_{(2)}\right|0\right\rangle$$

$$= \left\langle \pi^+\pi^-\eta \left| -\mathcal{J}^{\text{em}}_\mu \cdot \frac{-4ig_2}{\sqrt{6}M_V F^3}\epsilon_{\mu\nu\alpha\beta}\rho^{\mu\nu}\left[\partial^{\alpha\gamma}\partial^\beta\partial_\gamma(\cdots) - \partial_\gamma\partial^\beta\partial^{\alpha\gamma}(\cdots)\right] \cdot \frac{-eF_V}{2}\rho_{\theta\varphi}(\partial^\theta\mathscr{A}^\varphi - \partial^\varphi\mathscr{A}^\theta)\right| 0\right\rangle$$

$$= \frac{2ieF_V g_2\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_V F^3}\left\langle \pi^+\pi^-\eta \left| -\mathcal{J}^{\text{em}}_\mu \cdot \left[\partial^{\alpha\gamma}\partial^\beta\partial_\gamma(\cdots) - \partial_\gamma\partial^\beta\partial^{\alpha\gamma}(\cdots)\right] \cdot \frac{i(g^\mu_\theta g^\nu_\varphi - g^\nu_\theta g^\mu_\varphi)}{M^2_\rho - Q^2}(\partial^\theta\mathscr{A}^\varphi - \partial^\varphi\mathscr{A}^\theta)\right| 0\right\rangle$$

$$= \frac{-4eF_V g_2\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_V F^3(M^2_\rho - Q^2)}\left\langle \pi^+\pi^-\eta \left| -\mathcal{J}^{\text{em}}_\mu \cdot \left[\partial^{\alpha\gamma}\partial^\beta\partial_\gamma(\cdots) - \partial_\gamma\partial^\beta\partial^{\alpha\gamma}(\cdots)\right] \cdot (-2)\partial^\nu\mathscr{A}^\mu\right| 0\right\rangle$$

$$= \frac{8F_V g_2\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_V F^3(M^2_\rho - Q^2)}(-iQ^\nu)\left[\begin{pmatrix} 1^\alpha 1^\gamma 2^\beta 3_\gamma & & 2^\alpha 2^\gamma 3^\beta 1_\gamma \\ 3^\alpha 3^\gamma 1^\beta 2_\gamma & - & 2^\alpha 2^\gamma 1^\beta 3_\gamma \\ 1^\alpha 1^\gamma 3^\beta 2_\gamma & & 3^\alpha 3^\gamma 2^\beta 1_\gamma \end{pmatrix} - \begin{pmatrix} 1_\gamma 2^\beta 3^\alpha 3^\gamma & & 2_\gamma 3^\beta 1^\alpha 1^\gamma \\ 3_\gamma 1^\beta 2^\alpha 2^\gamma & - & 2_\gamma 1^\beta 3^\alpha 3^\gamma \\ 1_\gamma 3^\beta 2^\alpha 2^\gamma & & 3_\gamma 2^\beta 1^\alpha 1^\gamma \end{pmatrix}\right]$$

$$= \frac{-4\sqrt{6}iF_V g_2\epsilon_{\mu\nu\alpha\beta}}{3M_V F^3(M^2_\rho - Q^2)}1^\nu 2^\alpha 3^\beta(13 + 23 - 12 - 12 + 23 + 13 - 12 + 23 + 13 + 13 + 23 - 12)$$

$$= \left[\frac{8\sqrt{6}iF_V g_2\epsilon_{\mu\nu\alpha\beta}}{3M_V F^3(M^2_\rho - Q^2)}k^\nu_1 k^\alpha_2 k^\beta_3\right] \cdot 2(k_1k_2 - k_1k_3 - k_2k_3) = [\cdots] \cdot \left[(s - 2m^2_\pi) - (Q^2 - m^2_\eta - s)\right]$$

$$F^\eta_b(2) = \frac{8\sqrt{6}F_V}{3M_V F^3(M^2_\rho - Q^2)} \cdot g_2(2s - Q^2 + m^2_\eta - 2m^2_\pi)$$

15.2.3 $\mathcal{O}^3_{\text{VPPP}}$

$$\mathcal{L}^3_{\text{VPPP}} = \frac{g_3}{M_V}\mathcal{O}^3_{\text{VPPP}} = \frac{g_3}{M_V} \cdot i\epsilon_{\mu\nu\alpha\beta}\left\langle V^{\mu\nu}(u_\gamma h^{\alpha\gamma}u^\beta - u^\beta h^{\alpha\gamma}u_\gamma)\right\rangle$$

$$= \frac{g_3}{M_V} \cdot i\epsilon_{\mu\nu\alpha\beta}\left\langle V^{\mu\nu}\left[\left(-\frac{\sqrt{2}}{F}\right)\partial_\gamma\phi \cdot \left(-\frac{\sqrt{2}}{F}\right)(\partial^\alpha\partial^\gamma + \partial^\gamma\partial^\alpha)\phi \cdot \left(-\frac{\sqrt{2}}{F}\right)\partial^\beta\phi - \cdots\right]\right\rangle$$

$$= \frac{-2\sqrt{2}ig_3}{\sqrt{3}F^3 M_V}\epsilon_{\mu\nu\alpha\beta}\left[\rho^{\mu\nu}\partial_\gamma\pi^+(\partial^\alpha\partial^\gamma + \partial^\gamma\partial^\alpha)\pi^-\partial^\beta\eta - \rho^{\mu\nu}\partial^\beta\pi^+(\partial^\alpha\partial^\gamma + \partial^\gamma\partial^\alpha)\pi^-\partial_\gamma\eta\right]$$

$$T^\eta_\mu \equiv \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\rm em}_\mu e^{i\mathcal{L}_{\rm QCD}}\right|0\right\rangle \approx \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\rm em}_\mu \cdot i\mathcal{L}^3_{\rm VPPP}\cdot i\mathcal{L}^{\rm V}_{(2)}\right|0\right\rangle$$

$$= \left\langle \pi^+\pi^-\eta \left| -\mathcal{J}^{\rm em}_\mu \cdot \frac{-4ig_3}{\sqrt{6}M_V F^3}\epsilon_{\mu\nu\alpha\beta}\rho^{\mu\nu}\left[\partial_\gamma\partial^{\alpha\gamma}\partial^\beta(\cdots) - \partial^\beta\partial^{\alpha\gamma}\partial_\gamma(\cdots)\right]\cdot \frac{-eF_V}{2}\rho_{\theta\varphi}(\partial^\theta\mathscr{A}^\varphi - \partial^\varphi\mathscr{A}^\theta)\right| 0\right\rangle$$

$$= \frac{2ieF_V g_3\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_V F^3}\left\langle \pi^+\pi^-\eta\left|-\mathcal{J}^{\rm em}_\mu\cdot\left[\partial_\gamma\partial^{\alpha\gamma}\partial^\beta(\cdots) - \partial^\beta\partial^{\alpha\gamma}\partial_\gamma(\cdots)\right]\cdot\frac{i(g^\mu_\theta g^\nu_\varphi - g^\nu_\theta g^\mu_\varphi)}{M^2_\rho - Q^2}(\partial^\theta\mathscr{A}^\varphi - \partial^\varphi\mathscr{A}^\theta)\right|0\right\rangle$$

$$= \frac{-4eF_V g_3\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_V F^3(M^2_\rho - Q^2)}\left\langle \pi^+\pi^-\eta\left|-\mathcal{J}^{\rm em}_\mu\cdot\left[\partial_\gamma\partial^{\alpha\gamma}\partial^\beta(\cdots) - \partial^\beta\partial^{\alpha\gamma}\partial_\gamma(\cdots)\right]\cdot(-2)\partial^\nu\mathscr{A}^\mu\right|0\right\rangle$$

$$= \frac{8F_V g_3\epsilon_{\mu\nu\alpha\beta}}{\sqrt{6}M_V F^3(M^2_\rho - Q^2)}(-iQ^\nu)\left[\begin{pmatrix} 1_\gamma 2^\alpha 2^\gamma 3^\beta & & 2_\gamma 3^\alpha 3^\gamma 1^\beta \\ 3_\gamma 1^\alpha 1^\gamma 2^\beta & - & 2_\gamma 1^\alpha 1^\gamma 3^\beta \\ 1_\gamma 3^\alpha 3^\gamma 2^\beta & & 3_\gamma 2^\alpha 2^\gamma 1^\beta\end{pmatrix} - \begin{pmatrix} 1^\beta 2^\alpha 2^\gamma 3_\gamma & & 2^\beta 3^\alpha 3^\gamma 1_\gamma \\ 3^\beta 1^\alpha 1^\gamma 2_\gamma & - & 2^\beta 1^\alpha 1^\gamma 3_\gamma \\ 1^\beta 3^\alpha 3^\gamma 2_\gamma & & 3^\beta 2^\alpha 2^\gamma 1_\gamma\end{pmatrix}\right]$$

$$= \frac{-4\sqrt{6}iF_V g_1\epsilon_{\mu\nu\alpha\beta}}{3M_V F^3(M^2_\rho - Q^2)}1^\nu 2^\alpha 3^\beta(12+13-13-23+12+23+23+12-23-13+13+12)$$

$$= \left[\frac{-8\sqrt{6}iF_V g_3\epsilon_{\mu\nu\alpha\beta}}{3M_V F^3(M^2_\rho - Q^2)}k^\nu_1 k^\alpha_2 k^\beta_3\right]\cdot 2(k_1k_2) = [\cdots]\cdot(s - 2m^2_\pi)$$

$$F^\eta_b(3) = \frac{-8\sqrt{6}F_V}{3M_V F^3(M^2_\rho - Q^2)}\cdot g_3(s - 2m^2_\pi)$$

15.2.4 $\mathcal{O}^4_{\rm VPPP}$

$$\mathcal{O}^4_{\rm VPPP} = \epsilon_{\mu\nu\alpha\beta}\left\langle\left\{V^{\mu\nu}, u^\alpha u^\beta\right\}\chi_-\right\rangle$$

$$= \epsilon_{\mu\nu\alpha\beta}\left\langle\left[V^{\mu\nu}\left(-\frac{\sqrt{2}}{F}\right)\partial^\alpha\phi\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial^\beta\phi + \left(-\frac{\sqrt{2}}{F}\right)\partial^\alpha\phi\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial^\beta\phi V^{\mu\nu}\right]\cdot\frac{\sqrt{2}i}{F}(\phi\chi + \phi\chi)\right\rangle$$

$$= \frac{4\sqrt{2}i}{F^3}\epsilon_{\mu\nu\alpha\beta}\left\langle\left(V^{\mu\nu}\partial^\alpha\phi\partial^\beta\phi + \partial^\alpha\phi\partial^\beta\phi V^{\mu\nu}\right)\phi\right\rangle\cdot m^2_\pi$$

$$= \frac{4\sqrt{2}im^2_\pi}{\sqrt{3}F^3}\epsilon_{\mu\nu\alpha\beta}\rho^{\mu\nu}\partial^\alpha\pi^+\partial^\beta\pi^-\eta\times 2\left[\mu\nu\leftrightarrow\alpha\beta\right] = \frac{8\sqrt{2}im^2_\pi}{\sqrt{3}F^3}\epsilon_{\mu\nu\alpha\beta}\rho^{\mu\nu}\partial^\alpha\pi^+\partial^\beta\pi^-\eta$$

$$T^\eta_\mu \equiv \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\rm em}_\mu e^{i\mathcal{L}_{\rm QCD}}\right|0\right\rangle \approx \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\rm em}_\mu \cdot i\mathcal{L}^4_{\rm VPPP}\cdot i\mathcal{L}^{\rm V}_{(2)}\right|0\right\rangle$$

$$= \left\langle\pi^+\pi^-\eta\right|\mathcal{J}^{\rm em}_\mu\cdot\frac{8\sqrt{2}ig_4m^2_\pi}{\sqrt{3}M_V F^3}\epsilon_{\mu\nu\alpha\beta}\rho^{\mu\nu}\partial^\alpha\pi^+\partial^\beta\pi^-\eta\cdot\frac{eF_V}{2}\rho_{\mu\nu}(\partial^\mu\mathscr{A}^\nu - \partial^\nu\mathscr{A}^\mu)\left|0\right\rangle$$

$$= \frac{8\sqrt{2}iF_V g_4m^2_\pi}{\sqrt{3}M_V F^3}\epsilon_{\mu\nu\alpha\beta}\cdot ik^\alpha_1 ik^\beta_2\cdot\frac{-i}{M^2_\rho}\frac{1}{Q^2 - M^2_\rho}\left[g^\mu_\theta g^\nu_\varphi(M^2_\rho - Q^2) + g^\mu_\theta Q^\nu Q_\varphi - g^\mu_\varphi Q^\nu Q_\theta - (\mu\leftrightarrow\nu)\right]\cdot iQ^\varphi$$

$$= \frac{8\sqrt{2}iF_V g_4m^2_\pi}{\sqrt{3}M_V F^3}\epsilon_{\mu\nu\alpha\beta}\cdot\frac{1}{M^2_\rho - Q^2}k^\alpha_1 k^\beta_2\cdot 2(k_1 + k_2 + k_3)^\nu = \frac{8\sqrt{2}iF_V g_4m^2_\pi}{\sqrt{3}M_V F^3(M^2_\rho - Q^2)}\epsilon_{\mu\nu\alpha\beta}\cdot k^\nu_1 k^\alpha_2 k^\beta_3\cdot 2$$

$$F^\eta_b(4) = \frac{8\sqrt{6}F_V}{3M_V F^3(M^2_\rho - Q^2)}\cdot 2g_4m^2_\pi$$

15.2.5 $\mathcal{O}^5_{\text{VPPP}}$

$$\begin{aligned}\mathcal{L}^5_{\text{VPPP}} &= \frac{g_5}{M_V}\mathcal{O}^5_{\text{VPPP}} = \frac{g_5}{M_V}\cdot\epsilon_{\mu\nu\alpha\beta}\left\langle u^\alpha V^{\mu\nu}u^\beta\chi_-\right\rangle\\ &= \epsilon_{\mu\nu\alpha\beta}\left\langle\left[\left(-\frac{\sqrt{2}}{F}\right)\partial^\alpha\phi\cdot V^{\mu\nu}\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial^\beta\phi\right]\cdot\frac{\sqrt{2}i}{F}(\phi\chi+\phi\chi)\right\rangle\\ &= \frac{4\sqrt{2}i}{F^3}\epsilon_{\mu\nu\alpha\beta}\left\langle\left(\partial^\alpha\phi\cdot V^{\mu\nu}\cdot\partial^\beta\phi\right)\phi\right\rangle\cdot m_\pi^2 = \frac{4\sqrt{2}im_\pi^2}{\sqrt{3}F^3}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha\pi^+\cdot\rho^{\mu\nu}\cdot\partial^\beta\pi^-\cdot\eta\end{aligned}$$

$$\begin{aligned}T^\eta_\mu &\equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\text{em}}_\mu e^{i\mathcal{L}_{\text{QCD}}}\right|0\right\rangle \approx \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\text{em}}_\mu\cdot i\mathcal{L}^5_{\text{VPPP}}\cdot i\mathcal{L}^{\text{V}}_{(2)}\right|0\right\rangle\\ &= \left\langle\pi^+\pi^-\eta\right|\mathcal{J}^{\text{em}}_\mu\cdot\frac{4\sqrt{2}ig_4m_\pi^2}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\alpha\beta}\rho^{\mu\nu}\partial^\alpha\pi^+\partial^\beta\pi^-\eta\cdot\frac{eF_V}{2}\rho_{\mu\nu}(\partial^\mu\mathscr{A}^\nu-\partial^\nu\mathscr{A}^\mu)\left|0\right\rangle\\ &= \frac{4\sqrt{2}iF_Vg_4m_\pi^2}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\alpha\beta}\cdot ik_1^\alpha ik_2^\beta\cdot\frac{-i}{M_\rho^2}\frac{1}{Q^2-M_\rho^2}\left[g^\mu_\theta g^\nu_\varphi(M_\rho^2-Q^2)+g^\mu_\theta Q^\nu Q_\varphi-g^\mu_\varphi Q^\nu Q_\theta-(\mu\leftrightarrow\nu)\right]\cdot iQ^\varphi\\ &= \frac{4\sqrt{2}iF_Vg_4m_\pi^2}{\sqrt{3}M_VF^3}\cdot\frac{\epsilon_{\mu\nu\alpha\beta}}{M_\rho^2-Q^2}k_1^\alpha k_2^\beta\cdot 2(k_1+k_2+k_3)^\nu = \frac{8\sqrt{2}iF_Vg_4m_\pi^2}{\sqrt{3}M_VF^3(M_\rho^2-Q^2)}\epsilon_{\mu\nu\alpha\beta}\cdot k_1^\nu k_2^\alpha k_3^\beta\end{aligned}$$

$$F^\eta_b(5) = \frac{8\sqrt{6}F_V}{3M_VF^3(M_\rho^2-Q^2)}\cdot g_5m_\pi^2$$

15.3 VJP terms

$$\mathcal{L}^{\text{V}}_{(4)} = \sum_{j=1}^{7}\frac{c_j}{M_V}\mathcal{O}^j_{\text{VJP}} + \sum_{j=1}^{5}\frac{g_j}{M_V}\mathcal{O}^j_{\text{VPPP}}$$

$$\begin{aligned}&\mathcal{O}^1_{\text{VJP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu},f_+^{\rho\alpha}\}\nabla_\alpha u^\sigma\rangle && \mathcal{O}^2_{\text{VJP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\alpha},f_+^{\rho\sigma}\}\nabla_\alpha u^\nu\rangle\\ &\mathcal{O}^3_{\text{VJP}} = i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu},f_+^{\rho\sigma}\}\chi_-\rangle && \mathcal{O}^4_{\text{VJP}} = i\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}[f_-^{\rho\sigma},\chi_+]\rangle\\ &\mathcal{O}^5_{\text{VJP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu},f_+^{\rho\alpha}\}u^\sigma\rangle && \mathcal{O}^6_{\text{VJP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\alpha},f_+^{\rho\sigma}\}u^\nu\rangle\\ &\mathcal{O}^7_{\text{VJP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu},f_+^{\rho\alpha}\}u_\alpha\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{L}^{\text{V}}_{(2)} &= \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle + \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle \supset \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle = \frac{2iG_V}{F^2}\rho_{\mu\nu}\cdot\partial^\mu\pi^+\cdot\partial^\nu\pi^-\\ &= \frac{iG_V}{\sqrt{2}}\left\langle\frac{1}{\sqrt{2}}\begin{pmatrix}\rho & . & .\\ . & -\rho & .\\ . & . & 0\end{pmatrix}_{\mu\nu}\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial^\mu\begin{pmatrix}. & \pi^+ & .\\ \pi_- & . & .\\ . & . & 0\end{pmatrix}\cdot\left(-\frac{\sqrt{2}}{F}\right)\partial^\nu\begin{pmatrix}. & \pi^+ & .\\ \pi_- & . & .\\ . & . & 0\end{pmatrix}\right\rangle\end{aligned}$$

$$\begin{aligned}T^\eta_\mu &\equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\text{em}}_\mu e^{i\mathcal{L}_{\text{QCD}}}\right|0\right\rangle = \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\text{em}}_\mu e^{i\left(\mathcal{L}^{\text{V}}_{(2)}+\mathcal{L}_{\text{VJP}}\right)}\right|0\right\rangle\\ &\approx \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\text{em}}_\mu\left(1+i\mathcal{L}^{\text{V}}_{(2)}\right)(1+i\mathcal{L}_{\text{VJP}})\right|0\right\rangle\\ &= \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|-\mathcal{J}^{\text{em}}_\mu\cdot\mathcal{L}^{\text{V}}_{(2)}\cdot\mathcal{L}_{\text{VJP}}\right|0\right\rangle\end{aligned}$$

15.3.1 $\mathcal{O}^1_{\text{VJP}}$

$$\begin{aligned}
\mathcal{O}^1_{\text{VJP}} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\alpha}\}\,\nabla_\alpha u^\sigma\rangle \\
&\approx \epsilon_{\mu\nu\rho\sigma}\left\langle \{V^{\mu\nu}, 2eQ(\partial^\alpha\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\alpha)\}\,\nabla_\alpha\left(-\frac{\sqrt{2}\partial^\sigma\phi}{F}\right)\right\rangle \\
&\approx \epsilon_{\mu\nu\rho\sigma}\left\langle \{V^{\mu\nu}, Q\}\,\frac{2\sqrt{2}e}{F}(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\partial_\alpha\partial^\sigma\phi\right\rangle \\
&= \frac{2\sqrt{2}e}{F}\epsilon_{\mu\nu\rho\sigma}\left\langle\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}\rho & . & . \\ . & -\rho & . \\ . & . & 0\end{pmatrix}^{\mu\nu}, \begin{pmatrix}2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3\end{pmatrix}\right\}\right. \\
&\qquad\qquad \left.(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\partial_\alpha\partial^\sigma\frac{1}{\sqrt{6}}\begin{pmatrix}\eta & . & . \\ . & \eta & . \\ . & . & -2\eta\end{pmatrix}\right\rangle \\
&= \frac{4e}{\sqrt{6}F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\cdot\rho^{\mu\nu}\cdot\partial^\alpha\partial^\sigma\eta
\end{aligned}$$

$$\begin{aligned}
T^\eta_\mu &\equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\mathcal{J}^{\text{em}}_\mu e^{i\mathcal{L}_{\text{QCD}}}\middle|\,0\right\rangle \approx \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\mathcal{J}^{\text{em}}_\mu\cdot i\mathcal{L}^1_{\text{VJP}}\cdot i\mathcal{L}^{\text{V}}_{(2)}\middle|\,0\right\rangle \\
&= \left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\text{em}}_\mu\cdot\frac{4c_1e}{\sqrt{6}M_VF}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\rho^{\mu\nu}\partial_\alpha\partial^\sigma\eta\cdot\frac{2iG_V}{F^2}\rho_{\theta\varphi}\partial^\theta\pi^+\partial^\varphi\pi^-\,|0\rangle \\
&= \frac{8iG_Vc_1e}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\text{em}}_\mu\cdot(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\partial_\alpha\partial^\sigma\eta\cdot\frac{i}{M_\rho^2}\frac{1}{M_\rho^2-s} \\
&\qquad\left[g^\mu_\theta g^\nu_\varphi(M_\rho^2-s) + g^\mu_\theta(k_1+k_2)^\nu(k_1+k_2)_\varphi - g^\mu_\varphi(k_1+k_2)^\nu(k_1+k_2)_\theta - (\mu\to\nu)\right]\partial^\theta\pi^+\partial^\varphi\pi^-\,|0\rangle \\
&= \frac{8iG_Vc_1e}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\text{em}}_\mu\cdot(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\partial_\alpha\partial^\sigma\eta\cdot\frac{i}{M_\rho^2}\frac{1}{M_\rho^2-s} \\
&\qquad\left[\partial^\mu\pi^+\partial^\nu\pi^-(M_\rho^2-s) + \partial^\mu\pi^+(k_1+k_2)^\nu(k_1+k_2)_\varphi\partial^\varphi\pi^- - \partial^\mu\pi^-(k_1+k_2)^\nu(k_1+k_2)_\theta\partial^\theta\pi^+ - (\mu\to\right. \\
&= \frac{8iG_Vc_1e}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\text{em}}_\mu(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\partial_\alpha\partial^\sigma\eta\cdot\frac{i}{M_\rho^2-s}(\partial^\mu\pi^+\partial^\nu\pi^- - \partial^\nu\pi^+\partial^\mu\pi^-)\,|0\rangle \\
&= \frac{8iG_Vc_1e}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\text{em}}_\mu(\partial^\mu\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\mu)\partial_\alpha\partial^\sigma\eta\cdot\frac{2i}{M_\rho^2-s}\partial^\nu\pi^+\partial^\rho\pi^-\,|0\rangle \\
&= \frac{-16G_Vc_1}{\sqrt{6}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}\cdot iQ^\alpha\cdot ik_{3\alpha}ik_3^\sigma\cdot ik_1^\nu ik_2^\rho \\
&= \left[\frac{-8iG_Vc_1}{\sqrt{6}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma\right]\cdot 2Q^\alpha k_{3\alpha} = [\cdots]\cdot\left[Q^2 + k_3^2 - (Q-k_3)^2\right] \\
&= \frac{-4\sqrt{6}iG_Vc_1}{3M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma\cdot(Q^2+m_\eta^2-s)
\end{aligned}$$

$$F^\eta_c(1) = \frac{-4\sqrt{6}G_Vc_1}{3M_VF^3(M_\rho^2-s)}(Q^2+m_\eta^2-s)$$

15.3.2 $\mathcal{O}^2_{\rm VJP}$

$$\begin{aligned}
\mathcal{O}^2_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\alpha}, f_+^{\rho\sigma}\}\,\nabla_\alpha u^\nu\rangle \\
&= \epsilon_{\mu\nu\rho\sigma}\left\langle\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}\rho & . & . \\ . & -\rho & . \\ . & . & 0\end{pmatrix}^{\mu\alpha}, (-\frac{2e}{3})\begin{pmatrix}2 & . & . \\ . & -1 & . \\ . & . & -1\end{pmatrix}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\right\}\partial_\alpha\partial^\nu(-\frac{\sqrt{2}\phi}{F})\right\rangle \\
&= \frac{2\sqrt{2}e}{3\sqrt{3}F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\left\langle\begin{pmatrix}2\rho & . & . \\ . & \rho & . \\ . & . & 0\end{pmatrix}^{\mu\alpha}\partial_\alpha\partial^\nu\begin{pmatrix}\eta & . & . \\ . & \eta & . \\ . & . & -2\eta\end{pmatrix}\right\rangle \\
&= \frac{2\sqrt{2}e}{\sqrt{3}F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\rho^{\mu\alpha}\partial^\alpha\partial^\nu\eta
\end{aligned}$$

$$\begin{aligned}
T^\eta_\mu &\equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\mathcal{J}^{\rm em}_\mu e^{i\mathcal{L}_{\rm QCD}}\middle|\,0\right\rangle \approx \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\mathcal{J}^{\rm em}_\mu\cdot i\mathcal{L}^2_{\rm VJP}\cdot i\mathcal{L}^{\rm V}_{(2)}\middle|\,0\right\rangle \\
&= \left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\rm em}_\mu\cdot\frac{4\sqrt{2}c_2 e}{\sqrt{3}M_V F}\epsilon_{\mu\nu\rho\sigma}\partial^\rho\mathscr{A}^\sigma\rho^{\mu\alpha}\partial_\alpha\partial^\nu\eta\cdot\frac{2iG_V}{F^2}\rho_{\theta\varphi}\partial^\theta\pi^+\partial^\varphi\pi^-\,|0\rangle \\
&= \frac{8\sqrt{2}iG_V c_2 e}{\sqrt{3}M_V F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\rm em}_\mu\cdot\partial^\rho\mathscr{A}^\sigma\cdot\partial_\alpha\partial^\nu\eta\cdot\frac{i}{M_\rho^2}\frac{1}{M_\rho^2 - s} \\
&\qquad\left[g^\mu_\theta g^\alpha_\varphi(M_\rho^2 - s) + g^\mu_\theta(k_1+k_2)^\alpha(k_1+k_2)_\varphi - g^\mu_\varphi(k_1+k_2)^\alpha(k_1+k_2)_\theta - (\mu\to\alpha)\right]\partial^\theta\pi^+\partial^\varphi\pi^-\,|0\rangle \\
&= \frac{8\sqrt{2}iG_V c_2 e}{\sqrt{3}M_V F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\rm em}_\mu\cdot\partial^\rho\mathscr{A}^\mu\cdot\partial_\alpha\partial^\sigma\eta\cdot\frac{i}{M_\rho^2 - s}(\partial^\nu\pi^+\partial^\alpha\pi^- - \partial^\alpha\pi^+\partial^\nu\pi^-)\,|0\rangle \\
&= \frac{8\sqrt{2}iG_V c_2}{\sqrt{3}M_V F^3}\epsilon_{\mu\nu\rho\sigma}\cdot(-iQ^\rho)\cdot ik_{3\alpha}ik_3^\sigma\cdot\frac{i}{M_\rho^2 - s}(ik_1^\nu\cdot ik_2^\alpha - ik_1^\alpha\cdot ik_2^\nu) \\
&= \frac{8\sqrt{2}iG_V c_2}{\sqrt{3}M_V F^3(M_\rho^2 - s)}\epsilon_{\mu\nu\rho\sigma}(k_1+k_2+k_3)^\rho\cdot k_{3\alpha}k_3^\sigma(k_1^\nu k_2^\alpha - k_1^\alpha k_2^\nu) \\
&= \frac{8\sqrt{2}iG_V c_2}{\sqrt{3}M_V F^3(M_\rho^2 - s)}\epsilon_{\mu\nu\rho\sigma}\left[k_1^\nu(k_2^\alpha k_{3\alpha})k_2^\rho k_3^\sigma - (k_1^\alpha k_{3\alpha})k_2^\nu k_3^\sigma k_1^\rho\right] \\
&= \left[\frac{4\sqrt{2}iG_V c_2}{\sqrt{3}M_V F^3(M_\rho^2 - s)}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma\right]\cdot 2(k_2^\alpha k_{3\alpha} + k_1^\alpha k_{3\alpha}) = [\cdots]\cdot[2k_3(Q-k_3)] \\
&= [\cdots]\cdot\left[Q^2 - k_3^2 - (Q-k_3)^2\right] = \frac{4\sqrt{2}iG_V c_2}{\sqrt{3}M_V F^3(M_\rho^2 - s)}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma(Q^2 - m_\eta^2 - s)
\end{aligned}$$

$$F^\eta_c(2) = \frac{4\sqrt{6}G_V c_2}{3M_V F^3(M_\rho^2 - s)}(Q^2 - m_\eta^2 - s)$$

15.3.3 $\mathcal{O}^3_{\rm VJP}$

$$\chi_- = u^\dagger\chi u^\dagger - u\chi^\dagger u = -\frac{\sqrt{2}i}{F}(\phi\chi + \chi\phi)$$

$$\chi \equiv 2B_0(s+ip) \simeq 2B_0 s = M_\pi^2$$

$$\mathcal{O}^3_{\mathrm{VJP}} = i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\sigma}\}\,\chi_-\rangle$$

$$= i\epsilon_{\mu\nu\rho\sigma}\left\langle\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}\rho & \cdot & \cdot\\ \cdot & -\rho & \cdot\\ \cdot & \cdot & 0\end{pmatrix}^{\mu\nu}, \left(\frac{-2e}{3}\right)\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\right\}\right.$$

$$\left.\cdot\frac{-2\sqrt{2}iM_\pi^2}{F}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}\eta & \cdot & \cdot\\ \cdot & \eta & \cdot\\ \cdot & \cdot & -2\eta\end{pmatrix}\right\rangle$$

$$= \frac{-8eM_\pi^2}{\sqrt{6}F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\rho^{\mu\nu}\eta = \frac{-16eM_\pi^2}{\sqrt{6}F}\epsilon_{\mu\nu\rho\sigma}\partial^\rho\mathscr{A}^\sigma\rho^{\mu\nu}\eta$$

$$T_\mu^\eta \equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\mathrm{em}}e^{i\mathcal{L}_{\mathrm{QCD}}}\right|0\right\rangle \approx \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\mathrm{em}}\cdot i\mathcal{L}^3_{\mathrm{VJP}}\cdot i\mathcal{L}^{\mathrm{V}}_{(2)}\right|0\right\rangle$$

$$= \left\langle\pi^+\pi^-\eta\right| - \mathcal{J}_\mu^{\mathrm{em}}\cdot\frac{-16ec_3M_\pi^2}{\sqrt{6}M_VF}\epsilon_{\mu\nu\rho\sigma}\partial^\rho\mathscr{A}^\sigma\rho^{\mu\nu}\eta\cdot\frac{2iG_V}{F^2}\rho_{\theta\varphi}\partial^\theta\pi^+\partial^\varphi\pi^-\left|0\right\rangle$$

$$= \frac{-32iG_Vc_3M_\pi^2e}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}_\mu^{\mathrm{em}}\cdot\partial^\rho\mathscr{A}^\sigma\eta\cdot\frac{i}{M_\rho^2}\frac{1}{M_\rho^2-s}$$

$$\left[g_\theta^\mu g_\varphi^\nu(M_\rho^2-s) + g_\theta^\mu(k_1+k_2)^\nu(k_1+k_2)_\varphi - g_\varphi^\mu(k_1+k_2)^\nu(k_1+k_2)_\theta - (\mu\to\nu)\right]\partial^\theta\pi^+\partial^\varphi\pi^-\left|0\right\rangle$$

$$= \frac{-32iG_Vc_3M_\pi^2e}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}_\mu^{\mathrm{em}}\cdot\partial^\rho\mathscr{A}^\sigma\eta\cdot\frac{i}{M_\rho^2-s}(\partial^\mu\pi^+\partial^\nu\pi^- - \partial^\nu\pi^+\partial^\mu\pi^-)\left|0\right\rangle$$

$$= \frac{-32iG_Vc_3M_\pi^2e}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}_\mu^{\mathrm{em}}\cdot\partial^\nu\mathscr{A}^\mu\eta\frac{2i}{M_\rho^2-s}\cdot\partial^\sigma\pi^+\partial^\rho\pi^-\left|0\right\rangle$$

$$= \frac{-32iG_Vc_3M_\pi^2}{\sqrt{6}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\cdot(-iQ^\nu)\cdot\frac{2i}{M_\rho^2-s}\cdot ik_1^\sigma ik_2^\rho$$

$$= \frac{64G_Vc_3M_\pi^2}{\sqrt{6}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}\cdot(k_1+k_2+k_3)^\nu k_1^\sigma k_2^\rho$$

$$= \frac{-64G_Vc_3M_\pi^2}{\sqrt{6}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma$$

$$F_c^\eta(3) = \frac{-4\sqrt{6}G_V}{3M_VF^3(M_\rho^2-s)}\cdot 8c_3M_\pi^3$$

15.3.4 $\mathcal{O}^4_{\mathrm{VJP}}$

$$\mathcal{O}^4_{\mathrm{VJP}} = i\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}\,[f_-^{\rho\sigma}, \chi_+]\rangle$$

$$
\begin{aligned}
\mathcal{O}^4_{\mathrm{VJP}} &= i\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}\,[f_-^{\rho\sigma},\chi_+]\rangle \\
&\approx i\epsilon_{\mu\nu\rho\sigma}\left\langle V^{\mu\nu}\cdot\left[\frac{\sqrt{2}ie}{F}(\partial^\sigma\mathscr{A}^\rho-\partial^\rho\mathscr{A}^\sigma)\,[\phi,Q]\,,2\chi\right]\right\rangle \\
&= \frac{\sqrt{2}e}{F}\epsilon_{\mu\nu\rho\sigma}\,\langle V^{\mu\nu}\cdot[(\partial^\rho\mathscr{A}^\sigma-\partial^\sigma\mathscr{A}^\rho)(\phi Q-Q\phi),2\chi]\rangle \\
&= \frac{2\sqrt{2}e}{F}\epsilon_{\mu\nu\rho\sigma}\langle V^{\mu\nu}\,[(\phi Q-Q\phi),\chi]\rangle(\partial^\rho\mathscr{A}^\sigma-\partial^\sigma\mathscr{A}^\rho)=0
\end{aligned}
$$

15.3.5 $\mathcal{O}^5_{\mathrm{VJP}}$

$$
\begin{aligned}
\mathcal{O}^5_{\mathrm{VJP}} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu},f_+^{\rho\alpha}\}\,u^\sigma\rangle \\
&= \epsilon_{\mu\nu\rho\sigma}\left\langle\left\{\partial_\alpha\frac{1}{\sqrt{2}}\begin{pmatrix}\rho & \cdot & \cdot\\ \cdot & -\rho & \cdot\\ \cdot & \cdot & 0\end{pmatrix}_{\mu\nu}\cdot(-\frac{2e}{3})\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}(\partial^\rho\mathscr{A}^\alpha-\partial^\alpha\mathscr{A}^\rho)\right\}\cdot(-\frac{\sqrt{2}}{F})\partial^\sigma\phi\right\rangle \\
&= \frac{4e}{\sqrt{6}F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\alpha-\partial^\alpha\mathscr{A}^\rho)\partial_\alpha\rho^{\mu\nu}\cdot\partial^\sigma\eta
\end{aligned}
$$

$$
\begin{aligned}
T^\eta_\mu &\equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\mathcal{J}^{\mathrm{em}}_\mu e^{i\mathcal{L}_{\mathrm{QCD}}}\middle|\,0\right\rangle\approx\left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\mathcal{J}^{\mathrm{em}}_\mu\cdot i\mathcal{L}^5_{\mathrm{VPPP}}\cdot i\mathcal{L}^{\mathrm{V}}_{(2)}\middle|\,0\right\rangle \\
&= \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\right|-\mathcal{J}^{\mathrm{em}}_\mu\cdot\frac{4c_5e}{\sqrt{6}M_VF}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\alpha-\partial^\alpha\mathscr{A}^\rho)\partial_\alpha\rho^{\mu\nu}\partial^\sigma\eta\cdot\frac{2iG_V}{F^2}\rho_{\theta\varphi}\partial^\theta\pi^+\partial^\varphi\pi^-\,|0\rangle \\
&= \frac{4\sqrt{2}iG_Vc_5e}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right|-\mathcal{J}^{\mathrm{em}}_\mu(\partial^\rho\mathscr{A}^\alpha-\partial^\alpha\mathscr{A}^\rho)\partial^\sigma\eta\cdot\partial_\alpha\frac{i}{M_\rho^2-s}\left[g^\mu_\theta g^\nu_\varphi-(\mu\leftrightarrow\nu)\right]\partial^\theta\pi^+\partial^\varphi\pi^-\,|0\rangle \\
&= \frac{4\sqrt{2}iG_Vc_5e}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right|-\mathcal{J}^{\mathrm{em}}_\mu(\partial^\rho\mathscr{A}^\alpha-\partial^\alpha\mathscr{A}^\rho)\partial^\sigma\eta\cdot\partial_\alpha\frac{2i}{M_\rho^2-s}\partial^\mu\pi^+\partial^\nu\pi^-\,|0\rangle \\
&= \frac{4\sqrt{2}iG_Vc_5}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\rho\sigma}iQ^\alpha\cdot ik_3^\rho\cdot i(k_1+k_2)_\alpha\frac{2i}{M_\rho^2-s}\cdot ik_1^\sigma ik_2^\nu \\
&= \frac{-4\sqrt{2}iG_Vc_5}{\sqrt{3}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}(k_1+k_2+k_3)^\alpha k_3^\rho(k_1+k_2)_\alpha\cdot 2k_1^\sigma k_2^\nu \\
&= \left[\frac{-4\sqrt{2}iG_Vc_5}{\sqrt{3}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma\right]\cdot[2Q^\alpha(k_1+k_2)_\alpha]=[\cdots]\cdot[2Q(Q-k_3)] \\
&= [\cdots]\cdot\left[Q^2-k_3^2+(Q-k_3)^2\right]=\frac{-4\sqrt{2}iG_Vc_5}{\sqrt{3}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma\cdot(Q^2-m_\eta^2+s)
\end{aligned}
$$

$$
F^\eta_c(5)=\frac{-4\sqrt{6}G_Vc_5}{3M_VF^3(M_\rho^2-s)}\cdot(Q^2-m_\eta^2+s)
$$

15.3.6 $\mathcal{O}^6_{\mathrm{VJP}}$

$$\begin{aligned}
\mathcal{O}^6_{\mathrm{VJP}} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\alpha}, f_+^{\rho\sigma}\}\, u^\nu\rangle \\
&= \epsilon_{\mu\nu\rho\sigma}\left\langle\left\{\partial_\alpha \frac{1}{\sqrt{2}}\begin{pmatrix}\rho & . & . \\ . & -\rho & . \\ . & . & 0\end{pmatrix}^{\mu\nu}, (\frac{-2e}{3})\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0 \\ 0 & 0 & -1\end{pmatrix}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\right\}\frac{-\sqrt{2}}{F}\partial^\nu\phi\right\rangle \\
&= \frac{2e}{3\sqrt{6}F}\epsilon_{\mu\nu\rho\sigma}\left\langle 2\partial_\alpha\begin{pmatrix}\rho & . & . \\ . & -\rho & . \\ . & . & 0\end{pmatrix}^{\mu\nu}\cdot\begin{pmatrix}\eta & . & . \\ . & \eta & . \\ . & . & -2\eta\end{pmatrix}\right\rangle(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho) \\
&= \frac{8e}{\sqrt{6}F}\epsilon_{\mu\nu\rho\sigma}\cdot\partial_\alpha\rho^{\mu\alpha}\cdot\partial^\nu\eta\cdot\partial^\rho\mathscr{A}^\sigma
\end{aligned}$$

$$\begin{aligned}
T^\eta_\mu &\equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\mathrm{em}}_\mu e^{i\mathcal{L}_{\mathrm{QCD}}}\right|0\right\rangle \approx \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}^{\mathrm{em}}_\mu\cdot i\mathcal{L}^6_{\mathrm{VJP}}\cdot i\mathcal{L}^{\mathrm{V}}_{(2)}\right|0\right\rangle \\
&= \left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\mathrm{em}}_\mu\cdot\frac{8e}{\sqrt{6}F}\epsilon_{\mu\nu\rho\sigma}\partial_\alpha\rho^{\mu\alpha}\partial^\nu\eta\partial^\rho\mathscr{A}^\sigma\cdot\frac{2iG_V}{F^2}\rho_{\theta\varphi}\partial^\theta\pi^+\partial^\varphi\pi^-\left|0\right\rangle \\
&= \frac{8\sqrt{2}iG_Vc_6e}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\mathrm{em}}_\mu\cdot\partial^\rho\mathscr{A}^\sigma\cdot\partial^\nu\eta\cdot\partial_\alpha\rho^{\mu\alpha}\cdot\rho_{\theta\varphi}\partial^\theta\pi^+\partial^\varphi\pi^-\left|0\right\rangle \\
&= \frac{8\sqrt{2}iG_Vc_6e}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\mathrm{em}}_\mu\cdot\partial^\nu\mathscr{A}^\mu\cdot\partial^\rho\eta\cdot\partial_\alpha\rho^{\sigma\alpha}\cdot\rho_{\theta\varphi}\partial^\theta\pi^+\partial^\varphi\pi^-\left|0\right\rangle \\
&= \frac{8\sqrt{2}iG_Vc_6e}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\left\langle\pi^+\pi^-\eta\right| - \mathcal{J}^{\mathrm{em}}_\mu\cdot\partial^\nu\mathscr{A}^\mu\partial^\rho\eta\partial_\alpha\frac{i}{M_\rho^2-s}\left[g^\sigma_\theta g^\alpha_\varphi - (\sigma\leftrightarrow\alpha)\right]\partial^\theta\pi^+\partial^\varphi\pi^-\left|0\right\rangle \\
&= \frac{8\sqrt{2}iG_Vc_6}{\sqrt{3}M_VF^3}\epsilon_{\mu\nu\rho\sigma}\cdot(-iQ^\nu)\cdot ik_3^\rho\cdot i(k_1+k_2)_\alpha\cdot\frac{i}{M_\rho^2-s}(ik_1^\sigma ik_2^\alpha - ik_1^\alpha ik_2^\sigma) \\
&= \frac{8\sqrt{2}iG_Vc_6}{\sqrt{3}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}(k_1+k_2+k_3)^\nu k_3^\rho(k_1+k_2)_\alpha(k_1^\sigma k_2^\alpha - k_1^\alpha k_2^\sigma) \\
&= \frac{8\sqrt{2}iG_Vc_6}{\sqrt{3}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}\left[k_2^\nu k_3^\rho k_1^\sigma k_2^\alpha(k_1+k_2)_\alpha - k_1^\nu k_3^\rho k_2^\sigma k_1^\alpha(k_1+k_2)_\alpha\right] \\
&= \frac{8\sqrt{2}iG_Vc_6}{\sqrt{3}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}\left[k_1^\nu k_2^\rho k_3^\sigma\cdot k_2^\alpha(k_1+k_2)_\alpha + k_1^\nu k_2^\rho k_3^\sigma\cdot k_1^\alpha(k_1+k_2)_\alpha\right] \\
&= \frac{4\sqrt{2}iG_Vc_6}{\sqrt{3}M_VF^3(M_\rho^2-s)}\epsilon_{\mu\nu\rho\sigma}k_1^\nu k_2^\rho k_3^\sigma\cdot 2(k_1+k_2)^\alpha(k_1+k_2)_\alpha
\end{aligned}$$

$$F^\eta_c(6) = \frac{4\sqrt{6}G_V}{3M_VF^3(M_\rho^2-s)}\cdot 2c_6s$$

15.3.7 $\mathcal{O}^7_{\mathrm{VJP}}$

$$\mathcal{O}^7_{\mathrm{VJP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, f_+^{\rho\alpha}\}\, u_\alpha\rangle$$

$$\begin{aligned}\frac{C_7}{M_V}\mathcal{O}^7_{\mathrm{VJP}} &= \frac{C_7}{M_V}\epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, f_+^{\rho\alpha}\}\, u_\alpha\rangle \\ &\approx \frac{C_7}{M_V}\epsilon_{\mu\nu\rho\sigma}\left\langle\{\partial^\sigma V^{\mu\nu}, 2eQ(\partial^\alpha\mathscr{A}^\rho - \partial^\rho\mathscr{A}^\sigma)\}\left(-\frac{\sqrt{2}}{F}\partial_\alpha\phi\right)\right\rangle \\ &= \frac{2\sqrt{2}eC_7}{M_V F}\epsilon_{\mu\nu\rho\sigma}\langle\{\partial^\sigma V^{\mu\nu}, Q\}\,\partial_\alpha\phi\rangle(\partial^\rho\mathscr{A}^\sigma - \partial^\alpha\mathscr{A}^\rho) = 0\end{aligned}$$

15.4 VVP terms

$$\mathcal{O}^1_{\mathrm{VVP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\alpha}\}\,\nabla_\alpha u^\sigma\rangle \qquad \mathcal{O}^2_{\mathrm{VVP}} = i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\sigma}\}\,\chi_-\rangle$$

$$\mathcal{O}^3_{\mathrm{VVP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, V^{\rho\alpha}\}\,u^\sigma\rangle \qquad \mathcal{O}^4_{\mathrm{VVP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, V^{\rho\alpha}\}\,u_\alpha\rangle$$

$$\mathcal{L}^{\mathrm{V}}_{(2)} = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle$$

$$\frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle = \frac{-eF_V}{2}\rho_{\mu\nu}(\partial^\mu\mathscr{A}^\nu - \partial^\nu\mathscr{A}^\mu) \qquad \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle = \frac{2iG_V}{F^2}\rho_{\mu\nu}\cdot\partial^\mu\pi^+\cdot\partial^\nu\pi^-$$

$$\begin{aligned}T^\eta_\mu &\equiv \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\,\mathcal{J}^{\mathrm{em}}_\mu e^{i\mathcal{L}_{\mathrm{QCD}}}\,\middle|\,0\right\rangle = \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\,\mathcal{J}^{\mathrm{em}}_\mu e^{i\left(\mathcal{L}^{\mathrm{V}}_{(2)}+\mathcal{L}_{\mathrm{VVP}}\right)}\,\middle|\,0\right\rangle \\ &\approx \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\,\mathcal{J}^{\mathrm{em}}_\mu\left(1+i\cdot\frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle\right)(1+i\mathcal{L}_{\mathrm{VVP}})\left(1+i\cdot\frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle\right)\middle|\,0\right\rangle \\ &= \left\langle\pi^+(k_1)\pi^-(k_2)\eta(k_3)\,\middle|\,-\mathcal{J}^{\mathrm{em}}_\mu\cdot\frac{eG_VF_V}{F^2}\rho_{\theta\varphi}(\partial^\theta\mathscr{A}^\varphi - \partial^\varphi\mathscr{A}^\theta)\cdot d_j\mathcal{O}^j_{\mathrm{VVP}}\cdot\rho_{\alpha\beta}\partial^\alpha\pi^+\partial^\beta\pi^-\,\middle|\,0\right\rangle\end{aligned}$$

15.4.1 $\mathcal{O}^1_{\mathrm{VVP}}$

$$\begin{aligned}\mathcal{O}^1_{\mathrm{VVP}} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\alpha}\}\,\nabla_\alpha u^\sigma\rangle \\ &\approx \epsilon_{\mu\nu\rho\sigma}\left\langle\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}\rho & \cdot & \cdot\\ \cdot & -\rho & \cdot\\ \cdot & \cdot & 0\end{pmatrix}^{\mu\nu}, \frac{1}{\sqrt{2}}\begin{pmatrix}\rho & \cdot & \cdot\\ \cdot & -\rho & \cdot\\ \cdot & \cdot & 0\end{pmatrix}^{\rho\alpha}\right\}\partial_\alpha\left(\frac{-\sqrt{2}}{F}\right)\partial^\sigma\frac{1}{\sqrt{6}}\begin{pmatrix}\eta & \cdot & \cdot\\ \cdot & \eta & \cdot\\ \cdot & \cdot & -2\eta\end{pmatrix}\right\rangle \\ &= -\frac{\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F}(\rho^{\mu\nu}\rho^{\rho\alpha}\partial_\alpha\partial^\sigma\eta + \rho^{\rho\alpha}\rho^{\mu\nu}\partial_\alpha\partial^\sigma\eta)\end{aligned}$$

$$
\begin{aligned}
T_\mu^\eta &= \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3) \left| -\mathcal{J}_\mu^{\text{em}} \cdot \frac{eG_V F_V}{F^2}\rho_{\theta\varphi}(\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta)\cdot d_1 \mathcal{O}^1_{\text{VVP}} \cdot \rho_{\alpha\beta}\partial^\alpha\pi^+\partial^\beta\pi^- \right| 0 \right\rangle \\
&= \frac{-eG_V F_V d_1}{\sqrt{3}F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle \pi^\pm \eta \left| -\mathcal{J}_\mu^{\text{em}} \rho_{\theta\varphi}(\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta)(\rho^{\mu\nu}\rho^{\rho\alpha}\partial_\alpha\partial^\sigma\eta + \rho^{\rho\alpha}\rho^{\mu\nu}\partial_\alpha\partial^\sigma\eta)\rho_{\lambda\beta}\partial^\lambda\pi^+\partial^\beta\pi^- \right| 0\right\rangle \\
&= \frac{-eG_V F_V d_1}{\sqrt{3}F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle \pi^\pm\eta\right| - \mathcal{J}_\mu^{\text{em}} \cdot (\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta)\partial_\alpha\partial^\sigma\eta \cdot \left\{ \frac{i}{M_\rho^2 - Q^2}\left[g_\theta^\mu g_\varphi^\nu - (\mu \leftrightarrow \nu)\right] \cdot \right. \\
&\qquad \left. \frac{i}{M_\rho^2 - s}\left[g_\lambda^\rho g_\beta^\alpha - (\rho \leftrightarrow \alpha)\right] + [\mu\nu \leftrightarrow \rho\alpha] \right\} \partial^\lambda\pi^+\partial^\beta\pi^- \left|0\right\rangle + \{(\theta\varphi \sim \rho\alpha)(\mu\nu \sim \lambda\beta)\} \\
&= \frac{-eG_V F_V d_1}{\sqrt{3}F^3}\epsilon_{\mu\nu\rho\sigma}\left\langle \pi^\pm\eta\right| - \mathcal{J}_\mu^{\text{em}} \cdot \partial_\alpha\partial^\sigma\eta \cdot \left\{ \frac{2i(\partial^\mu \mathscr{A}^\nu - \partial^\nu \mathscr{A}^\mu)}{M_\rho^2 - Q^2} \cdot \frac{i(\partial^\rho\pi^+\partial^\alpha\pi^- - \partial^\alpha\pi^+\partial^\rho\pi^-)}{M_\rho^2 - s} \right. \\
&\qquad \left. + \frac{2i(\partial^\rho \mathscr{A}^\alpha - \partial^\alpha \mathscr{A}^\rho)}{M_\rho^2 - Q^2} \cdot \frac{i(\partial^\mu\pi^+\partial^\nu\pi^- - \partial^\nu\pi^+\partial^\mu\pi^-)}{M_\rho^2 - s} \right\} \left|0\right\rangle + \{(\theta\varphi \sim \rho\alpha)(\mu\nu \sim \lambda\beta)\} \\
&= \frac{-4G_V F_V d_1 e \epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}\left\langle \pi^\pm\eta\right| - \mathcal{J}_\mu^{\text{em}} \cdot \partial_\alpha\partial^\sigma\eta\left[\partial^\nu \mathscr{A}^\mu(\partial^\rho\pi^+\partial^\alpha\pi^- - \partial^\alpha\pi^+\partial^\rho\pi^-)\right. \\
&\qquad \left. + \partial^\alpha \mathscr{A}^\mu \cdot \partial^\nu\pi^+\partial^\rho\pi^-\right]\left|0\right\rangle + \{(\theta\varphi \sim \rho\alpha)(\mu\nu \sim \lambda\beta)\} \\
&= \frac{-4G_V F_V d_1 \epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}(ik_{3\alpha}ik_3^\sigma)\left[(-iQ^\nu)(ik_1^\rho ik_2^\alpha - ik_1^\alpha ik_2^\rho) + (-iQ^\alpha)ik_1^\nu ik_2^\rho\right] + \{\cdots\} \\
&= \frac{4iG_V F_V d_1 \epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}k_{3\alpha}k_3^\sigma\left[(k_1 + k_2 + k_3)^\nu(k_1^\rho k_2^\alpha - k_1^\alpha k_2^\rho) + (k_1 + k_2 + k_3)^\alpha k_1^\nu k_2^\rho\right] + \{\cdots\} \\
&= \frac{4iG_V F_V d_1 \epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}\left[k_{3\alpha}k_3^\sigma(k_2^\nu k_1^\rho k_2^\alpha - k_1^\nu k_1^\alpha k_2^\rho) + k_{3\alpha}k_3^\sigma k_1^\nu k_2^\rho Q^\alpha\right] + \{\cdots\} \\
&= \frac{4iG_V F_V d_1 \epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}k_1^\nu k_2^\rho k_3^\sigma\left(Q^\alpha k_{3\alpha} - k_1^\alpha k_{3\alpha} - k_2^\alpha k_{3\alpha}\right) + \{\cdots\} \\
&= \frac{8\sqrt{3}iG_V F_V d_1 \epsilon_{\mu\nu\rho\sigma}}{3F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}k_1^\nu k_2^\rho k_3^\sigma \cdot k_3^\alpha k_{3\alpha} = \frac{8\sqrt{3}iG_V F_V d_1 \epsilon_{\mu\nu\rho\sigma}}{3F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}k_1^\nu k_2^\rho k_3^\sigma \cdot m_\eta^2
\end{aligned}
$$

$$
F_d^\eta(1) = \frac{8\sqrt{3}G_V F_V d_1 m_\eta^2}{3F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}
$$

15.4.2 $\mathcal{O}^2_{\text{VVP}}$

$$
\begin{aligned}
\mathcal{O}^2_{\text{VVP}} &= i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\sigma}\}\,\chi_-\rangle \\
&= i\epsilon_{\mu\nu\rho\sigma}\left\langle \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} \rho & \cdot & \cdot \\ \cdot & -\rho & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}^{\mu\nu}, \frac{1}{\sqrt{2}}\begin{pmatrix} \rho & \cdot & \cdot \\ \cdot & -\rho & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}^{\rho\sigma} \right\} \left(\frac{-\sqrt{2}i}{F}\right) 2M_\pi^2 \cdot \frac{1}{\sqrt{6}}\begin{pmatrix} \eta & \cdot & \cdot \\ \cdot & \eta & \cdot \\ \cdot & \cdot & -2\eta \end{pmatrix} \right\rangle \\
&= \frac{2M_\pi^2}{\sqrt{3}F}\epsilon_{\mu\nu\rho\sigma}\left(\rho^{\mu\nu}\rho^{\rho\sigma}\eta + \rho^{\rho\sigma}\rho^{\mu\nu}\eta\right)
\end{aligned}
$$

$$
\begin{aligned}
T^{\eta}_{\mu} &= \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3) \left| -\mathcal{J}^{\text{em}}_{\mu} \cdot \frac{eG_V F_V}{F^2} \rho_{\theta\varphi}(\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta) \cdot d_2 \mathcal{O}^2_{\text{VVP}} \cdot \rho_{\alpha\beta} \partial^\alpha \pi^+ \partial^\beta \pi^- \right| 0 \right\rangle \\
&= \left\langle \pi^+\pi^-\eta \left| -\mathcal{J}^{\text{em}}_{\mu} \cdot \frac{eG_V F_V}{F^2} \rho_{\theta\varphi}(\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta) \cdot \frac{2d_2 M_\pi^2}{\sqrt{3}F} \epsilon_{\mu\nu\rho\sigma} \left(\rho^{\mu\nu}\rho^{\rho\sigma}\eta + \rho^{\rho\sigma}\rho^{\mu\nu}\eta\right) \cdot \rho_{\alpha\beta} \partial^\alpha \pi^+ \partial^\beta \pi^- \right| 0 \right\rangle \\
&= \frac{2eG_V F_V d_2 M_\pi^2}{\sqrt{3}F^3} \epsilon_{\mu\nu\rho\sigma} \left\langle \pi^+\pi^-\eta \right| - \mathcal{J}^{\text{em}}_{\mu} \cdot (\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta) \left\{ \frac{i}{M_\rho^2 - Q^2} \left[g^\mu_\theta g^\nu_\varphi - (\mu \leftrightarrow \nu) \right] \cdot \right. \\
&\qquad \left. \frac{i}{M_\rho^2 - s} \left[g^\rho_\alpha g^\sigma_\beta - (\rho \leftrightarrow \sigma) \right] + [\mu\nu \leftrightarrow \rho\sigma] \right\} \eta \partial^\alpha \pi^+ \partial^\beta \pi^- \left|0\right\rangle + \{(\theta\varphi \sim \rho\sigma)(\mu\nu \sim \alpha\beta)\} \\
&= \frac{2eG_V F_V d_2 M_\pi^2}{\sqrt{3}F^3} \epsilon_{\mu\nu\rho\sigma} \left\langle \pi^+\pi^-\eta \right| - \mathcal{J}^{\text{em}}_{\mu} \cdot \frac{-4i}{M_\rho^2 - Q^2} \partial^\nu \mathscr{A}^\mu \cdot \eta \cdot \frac{2i}{M_\rho^2 - s} \partial^\rho \pi^+ \partial^\sigma \pi^- \times 2 \left[\mu\nu \leftrightarrow \rho\sigma\right] \times 2 \left\{ \cdots \right\} \\
&= \frac{64 G_V F_V d_2 M_\pi^2}{\sqrt{3}F^3 (M_\rho^2 - Q^2)(M_\rho^2 - s)} \epsilon_{\mu\nu\rho\sigma} (-iQ^\nu) i k_1^\rho i k_2^\sigma = \frac{64\sqrt{3} i G_V F_V d_2 M_\pi^2 \epsilon_{\mu\nu\rho\sigma}}{3F^3 (M_\rho^2 - Q^2)(M_\rho^2 - s)} k_1^\nu k_2^\rho k_3^\sigma
\end{aligned}
$$

$$
F^\eta_d(2) = \frac{64\sqrt{3} G_V F_V d_2 M_\pi^2}{3F^3 (M_\rho^2 - Q^2)(M_\rho^2 - s)}
$$

15.4.3 $\mathcal{O}^3_{\text{VVP}}$

$$
\begin{aligned}
\mathcal{O}^3_{\text{VVP}} &= \epsilon_{\mu\nu\rho\sigma} \langle \{ \nabla_\alpha V^{\mu\nu}, V^{\rho\alpha} \} u^\sigma \rangle \\
&\approx \epsilon_{\mu\nu\rho\sigma} \left\langle \left\{ \partial_\alpha \frac{1}{\sqrt{2}} \begin{pmatrix} \rho & . & . \\ . & -\rho & . \\ . & . & 0 \end{pmatrix}^{\mu\nu}, \frac{1}{\sqrt{2}} \begin{pmatrix} \rho & . & . \\ . & -\rho & . \\ . & . & 0 \end{pmatrix}^{\rho\alpha} \right\} \left(\frac{-\sqrt{2}}{F}\right) \partial^\sigma \frac{1}{\sqrt{6}} \begin{pmatrix} \eta & . & . \\ . & \eta & . \\ . & . & -2\eta \end{pmatrix} \right\rangle \\
&= \frac{-\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F} \left(\partial_\alpha \rho^{\mu\nu} \cdot \rho^{\rho\alpha} \cdot \partial^\sigma \eta + \rho^{\rho\alpha} \cdot \partial_\alpha \rho^{\mu\nu} \cdot \partial^\sigma \eta \right)
\end{aligned}
$$

$$T_\mu^\eta = \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3) \left| -\mathcal{J}_\mu^{\text{em}} \cdot \frac{eG_VF_V}{F^2}\rho_{\theta\varphi}(\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta) \cdot d_3 \mathcal{O}^3_{\text{VVP}} \cdot \rho_{\lambda\beta}\partial^\lambda\pi^+\partial^\beta\pi^- \right| 0 \right\rangle$$
$$= \langle \pi^+\pi^-\eta | -\mathcal{J}_\mu^{\text{em}} \frac{eG_VF_V}{F^2}\rho_{\theta\varphi}(\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta)\frac{-d_3\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F}(\partial_\alpha\rho^{\mu\nu}\rho^{\rho\alpha}\partial^\sigma\eta + \rho^{\rho\alpha}\partial_\alpha\rho^{\mu\nu}\partial^\sigma\eta)\cdot\rho_{\lambda\beta}\partial^\lambda\pi^+\partial^\beta\pi^- |0$$
$$= \frac{-d_3eG_VF_V}{\sqrt{3}F^3}\epsilon_{\mu\nu\rho\sigma}\langle\pi^+\pi^-\eta| -\mathcal{J}_\mu^{\text{em}}(\partial^\theta \mathscr{A}^\varphi - \partial^\varphi \mathscr{A}^\theta)\cdot\left\{\left[\partial_\alpha\frac{i\left(g_\theta^\mu g_\varphi^\nu - g_\varphi^\mu g_\theta^\nu\right)}{M_\rho^2-Q^2}\cdot\frac{i(g_\lambda^\rho g_\beta^\alpha - g_\beta^\rho g_\lambda^\alpha)}{M_\rho^2-s}\right]\right.$$
$$\left. + \left[\frac{i\left(g_\theta^\rho g_\varphi^\alpha - g_\varphi^\rho g_\theta^\alpha\right)}{M_\rho^2-Q^2}\cdot\partial_\alpha\frac{i(g_\lambda^\mu g_\beta^\nu - g_\beta^\mu g_\lambda^\nu)}{M_\rho^2-s}\right]\right\}\partial^\sigma\eta\cdot\partial^\lambda\pi^+\partial^\beta\pi^-|0\rangle + \{(\mu\nu\sim\lambda\beta)(\rho\alpha\sim\theta\varphi)\}$$
$$= \frac{-d_3eG_VF_V}{\sqrt{3}F^3}\epsilon_{\mu\nu\rho\sigma}\langle\pi^+\pi^-\eta| -\mathcal{J}_\mu^{\text{em}}\left\{(\partial_\alpha)\frac{-4i\partial^\nu\mathscr{A}^\mu}{M_\rho^2-Q^2}\cdot\frac{i(\partial^\rho\pi^+\partial^\alpha\pi^- - \partial^\alpha\pi^+\partial^\rho\pi^-)}{M_\rho^2-s}\right.$$
$$\left. + \frac{2i(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)}{M_\rho^2-Q^2}\cdot(\partial_\alpha)\frac{2i\partial^\mu\pi^+\partial^\nu\pi^-}{M_\rho^2-s}\right\}\partial^\sigma\eta|0\rangle + \{(\mu\nu\sim\lambda\beta)(\rho\alpha\sim\theta\varphi)\}$$
$$= \frac{4d_3eG_VF_V\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2-Q^2)(M_\rho^2-s)}\langle\pi^\pm\eta|\mathcal{J}_\mu^{\text{em}}\left[(\partial_\alpha)\partial^\nu\mathscr{A}^\mu(\partial^\rho\pi^+\partial^\alpha\pi^- - \partial^\alpha\pi^+\partial^\rho\pi^-)\right.$$
$$\left. + \partial^\alpha\mathscr{A}^\mu(\partial_\alpha)\partial^\nu\pi^+\partial^\rho\pi^-\right]\partial^\sigma\eta|0\rangle + \{\cdots\}$$
$$= \frac{-4d_3G_VF_V\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2-Q^2)(M_\rho^2-s)}\left[(-iQ_\alpha)(-iQ^\nu)\cdot(ik_1^\rho ik_2^\alpha - ik_1^\alpha ik_2^\rho) + (-iQ^\alpha)i(k_1+k_2)_\alpha ik_1^\nu ik_2^\rho\right]ik_3^\sigma + \{\cdots\}$$
$$= \frac{-4id_3G_VF_V\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2-Q^2)(M_\rho^2-s)}\left(k_1^\rho k_2^\nu k_3^\sigma Q_\alpha k_2^\alpha - k_1^\nu k_2^\rho k_3^\sigma Q_\alpha k_1^\alpha - k_1^\nu k_2^\rho k_3^\sigma Q^\alpha(k_1+k_2)_\alpha\right) + \{\cdots\}$$
$$= \frac{4id_3G_VF_V\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2-Q^2)(M_\rho^2-s)}k_1^\nu k_2^\rho k_3^\sigma\left[Q_\alpha k_1^\alpha + Q_\alpha k_2^\alpha + (k_1+k_2)_\alpha Q^\alpha\right] + \{\cdots\}$$
$$= \left[\frac{8id_3G_VF_V\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2-Q^2)(M_\rho^2-s)}k_1^\nu k_2^\rho k_3^\sigma\right]\cdot 2Q(k_1+k_2) = [\cdots]\cdot 2Q(Q-k_3)$$
$$= [\cdots]\cdot\left[Q^2 - k_3^2 - (Q-k_3)^2\right] = \frac{8id_3G_VF_V\epsilon_{\mu\nu\rho\sigma}}{\sqrt{3}F^3(M_\rho^2-Q^2)(M_\rho^2-s)}k_1^\nu k_2^\rho k_3^\sigma\cdot(Q^2 - m_\eta^2 + s)$$

$$F_d^\eta(3) = \frac{8\sqrt{3}d_3G_VF_V}{3F^3(M_\rho^2-Q^2)(M_\rho^2-s)}\cdot(Q^2 - m_\eta^2 + s)$$

15.4.4 $\mathcal{O}^4_{\text{VVP}}$

$$\mathcal{O}^4_{\text{VVP}} = \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla^\sigma V^{\mu\nu}, V^{\rho\alpha}\}u_\alpha\rangle$$
$$\approx \epsilon_{\mu\nu\rho\sigma}\left\langle\{\partial^\sigma V^{\mu\nu}, V^{\rho\alpha}\}\left(-\frac{\sqrt{2}}{F}\partial_\alpha\phi\right)\right\rangle = 0$$

15.5 Brief summary

$$F_a^\eta = \frac{-N_C}{12\sqrt{3}\pi^2 F^3}$$

$$F_b^\eta = \frac{8\sqrt{6}F_V}{3M_V F^3(M_\rho^2 - Q^2)}\left[(g_1 - g_3)(s - 2m_\pi^2) + g_2(2s - Q^2 + m_\eta^2 - 2m_\pi^2) + (2g_4 + g_5)m_\pi^2\right]$$

$$F_c^\eta = \frac{-4\sqrt{6}G_V}{3M_V F^3(M_\rho^2 - s)}\left[c_1(Q^2 + m_\eta^2 - s) - c_2(Q^2 - m_\eta^2 - s) + 8c_3 m_\pi^2 + c_5(Q^2 - m_\eta^2 + s) - 2c_6 s\right]$$

$$F_d^\eta = \frac{8\sqrt{3}G_V F_V}{3F^3(M_\rho^2 - Q^2)(M_\rho^2 - s)}\left[d_1 m_\eta^2 + 8d_2 m_\pi^2 + d_3(Q^2 - m_\eta^2 + s)\right]$$

16 Calculations w/ mixing angles

Previously, form factors are calculated without considering mixing angles, which is quite ideal but not compatible with reality. Indeed, there are three mixing modes that we would take into account in the following mention, including:
$\eta - \eta'$ mixing,

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P \end{pmatrix}\begin{pmatrix} \eta_8 \\ \eta_0 \end{pmatrix} \qquad \Rightarrow \qquad \eta = \cos\theta_P \eta_8 - \sin\theta_P \eta_0$$

$\rho - \omega$ mixing,

$$\begin{pmatrix} \bar{\rho}^0 \\ \bar{\omega} \end{pmatrix} = \begin{pmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\theta \end{pmatrix}\begin{pmatrix} \rho^0 \\ \omega \end{pmatrix}$$

and $\omega_8 - \omega_0$ mixing

$$\begin{pmatrix} V_\mu^8 \\ V_\mu^0 \end{pmatrix} = \begin{pmatrix} \cos\theta_V & -\sin\theta_V \\ \sin\theta_V & \cos\theta_V \end{pmatrix}\begin{pmatrix} \phi_\mu \\ \omega_\mu \end{pmatrix} \qquad \Rightarrow \qquad \begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \cos\theta_V & -\sin\theta_V \\ \sin\theta_V & \cos\theta_V \end{pmatrix}\begin{pmatrix} \omega_8 \\ \omega_0 \end{pmatrix}$$

Therefore, the propagators in the last section should be re-written as

$$\rho \sim \bar{\rho}^0 = \cos\delta\rho^0 + \sin\delta\left(\sin\theta_V\omega_8 + \cos\theta_V\omega_0\right)$$

$$\omega \sim \bar{\omega} = -\sin\delta\rho^0 + \cos\delta\left(\sin\theta_V\omega_8 + \cos\theta_V\omega_0\right)$$

16.1 F_a^η

Note: SU(3) is replace by U(3),

$$\Phi \to \Phi + \eta_0/\sqrt{3}\cdot\mathbf{1} = \begin{pmatrix} \eta_8/\sqrt{6} + \eta_0/\sqrt{3} & \cdot & \cdot \\ \cdot & \eta_8/\sqrt{6} + \eta_0/\sqrt{3} & \cdot \\ \cdot & \cdot & \eta_0/\sqrt{3} - 2\eta_8/\sqrt{6} \end{pmatrix}$$

$$\begin{aligned} T_\mu^\eta &\equiv \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\text{em}} e^{i\mathcal{L}_{\text{QCD}}}\right|0\right\rangle \approx \left\langle \pi^+\pi^-\eta\left|\mathcal{J}_\mu^{\text{em}}\cdot i\mathcal{L}_{\text{QCD}}\right|0\right\rangle \\ &= \left\langle \pi^+\pi^-\left(\cos\theta_P\eta_8 - \sin\theta_P\eta_0\right)\left|\cdots\right|0\right\rangle = \cos\theta_P\left\langle \pi^+\pi^-\eta_8\left|\cdots\right|0\right\rangle - \sin\theta_P\left\langle \pi^+\pi^-\eta_0\left|\cdots\right|0\right\rangle \end{aligned}$$

Previously, we got $F_a^\eta = -\frac{N_C}{12\sqrt{3}\pi^2 F^3}$ (from $\eta_8/\sqrt{6}$). Thus for $\eta_0/\sqrt{3}$ we have to add a coefficient $\sqrt{2}$, then writing the following result:

$$F_a^\eta = -\frac{N_C}{12\sqrt{3}\pi^2 F^3}(\cos\theta_P - \sqrt{2}\sin\theta_P)$$

16.2 VPPP, F_b^η

For VPPP terms, review the form factor without mixing angles at first:

$$F_b^\eta = \frac{8\sqrt{6}F_V}{3M_V F^3(M_\rho^2 - Q^2)}\left[(g_1 - g_3)(s - 2m_\pi^2) + g_2(2s - Q^2 + m_\eta^2 - 2m_\pi^2) + (2g_4 + g_5)m_\pi^2\right]$$

Herein, other mixing should be taken into consideration apart from the $\eta - \eta'$ one, due to the propagator ρ. Introduce new parameter:

$$G_{R\eta} = (Q^2, s) = (g_1 + 2g_2 - g_3)(s - 2m_\pi^2) + g_2(-Q^2 + 2m_\pi^2 + m_\eta^2) + (2g_4 + g_5)m_\pi^2$$

as well as the modified F_V, for $\rho^0(770)$ and $\omega(782)$, proportional to pseudo=scalar masses,

$$F_V \to F_V\left(1 + \frac{8\sqrt{2}\alpha_V m_\pi^2}{M_V^2}\right)$$

and that for $\phi(1020)$

$$F_V \to F_V\left(1 + \frac{8\sqrt{2}\alpha_V(2m_K^2 - m_\pi^2)}{M_V^2}\right)$$

Also, we introduce widths of intermediate mesons at the same time. (Breit Wigner formula)

$$BW\left[V, x\right] = \frac{1}{M_V^2 - i\Gamma_V(x)M_V - x}$$

furthermore, up to around $E \sim 2\text{GeV}$, we have to include heavier resonances, including $\rho(1450)(V')$ and $\rho(1700)(V'')$, with the following substitutions in the form factor

$$BW_R\left[P, V, x\right] = BW\left[V, x\right] + \beta'_P BW\left[V', x\right] + \beta''_P BW\left[V'', x\right]$$

where β' and β'' are unknown real parameters related to the strength of the coupling of the V' and V'' multiplets. Thus

$$\frac{1}{M_\rho^2 - Q^2} \to BW_R\left[\eta, \rho, Q^2\right]$$

$$T_\mu^\eta \equiv \left\langle \pi^+(k_1)\pi^-(k_2)\eta(k_3)\left|\mathcal{J}_\mu^{\text{em}} e^{i\mathcal{L}_{\text{QCD}}}\right|0\right\rangle \approx \left\langle \pi^+\pi^-\eta\left|\mathcal{J}_\mu^{\text{em}} \cdot i\mathcal{L}^1_{\text{VPPP}} \cdot i\mathcal{L}^{\text{V}}_{(2)}\right|0\right\rangle$$

and for the latter $\mathcal{L}^{\text{V}}_{(2)} = \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle + \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle \supset \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu}f_+^{\mu\nu}\rangle$.

$$\frac{F_V}{2\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} \rho & \cdot & \cdot \\ \cdot & -\rho & \cdot \\ \cdot & \cdot & 0 \end{pmatrix}_{\mu\nu} \cdot \left(\frac{-2e}{3}\right)\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}(\partial^\mu\mathscr{A}^\nu - \partial^\nu\mathscr{A}^\mu)\right\rangle = -\frac{eF_V}{2}\rho_{\mu\nu}(\partial^\mu\mathscr{A}^\nu - \partial^\nu\mathscr{A}^\mu)$$

$$\frac{F_V}{2\sqrt{2}}\left\langle \frac{1}{\sqrt{3}}\begin{pmatrix} \omega_0 & . & . \\ . & \omega_0 & . \\ . & . & \omega_0 \end{pmatrix}_{\mu\nu} \cdot \left(\frac{-2e}{3}\right)\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}(\partial^\mu \mathscr{A}^\nu - \partial^\nu \mathscr{A}^\mu)\right\rangle = 0$$

$$\frac{F_V}{2\sqrt{2}}\left\langle \frac{1}{\sqrt{6}}\begin{pmatrix} \omega_8 & . & . \\ . & \omega_8 & . \\ . & . & -2\omega_8 \end{pmatrix}_{\mu\nu} \cdot \left(\frac{-2e}{3}\right)\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}(\partial^\mu \mathscr{A}^\nu - \partial^\nu \mathscr{A}^\mu)\right\rangle = -\frac{eF_V}{2\sqrt{3}}(\omega_8)_{\mu\nu}(\partial^\mu \mathscr{A}^\nu - \partial^\nu$$

However, only ρ^0 would survive for VPPP terms. So for the propagator with ρ^0 at the beginning

$$\bar{\rho}^0 \sim \cos\delta\rho^0 + \sin\delta\sin\theta_V\omega_8 \qquad \bar{\omega} = -\sin\delta\rho^0 + \cos\delta\sin\theta_V\omega_8$$

but in the end

$$\bar{\rho}^0 \sim \cos\delta\rho^0 \qquad \bar{\omega} \sim -\sin\delta\rho^0$$

Finally, with $\eta - \eta'$ mixing mentioned above, we combine and furnish new form factor for VPPP

$$F_b^\eta = \frac{8\sqrt{6}F_V\left(1 + \frac{8\sqrt{2}\alpha_V m_\pi^2}{M_V^2}\right)}{3M_V F^3} G_{R\eta}(Q^2, s)\left\{BW_R\left[\eta, \rho, Q^2\right] \cdot \cos\delta(\cos\delta + \frac{1}{\sqrt{3}}\sin\delta\sin\theta_V)\right.$$
$$\left. + BW_R\left[\eta, \omega, Q^2\right] \cdot (-\sin\delta)(-\sin\delta + \frac{1}{\sqrt{3}}\cos\delta\sin\theta_V)\right\}(\cos\theta_P - \sqrt{2}\sin\theta_P)$$

16.3 VJP, F_c^η

$$\begin{aligned}
\rho\eta \to & \left[\cos\delta\rho^0 + \sin\delta\left(\sin\theta_V\omega_8 + \cos\theta_V\omega_0\right)\right](\cos\theta_P\eta_8 - \sin\theta_P\eta_0) \\
&= \cos\delta\cos\theta_P \cdot \rho^0\eta_8 - \cos\delta\sin\theta_P \cdot \rho^0\eta_0 + \sin\delta\sin\theta_V\cos\theta_P \cdot \omega_8\eta_8 \\
&\quad - \sin\delta\sin\theta_V\sin\theta_P \cdot \omega_8\eta_0 + \sin\delta\cos\theta_V\cos\theta_P \cdot \omega_0\eta_8 - \sin\delta\cos\theta_V\sin\theta_P \cdot \omega_0\eta_0
\end{aligned}$$

$$\begin{aligned}
\omega\eta \to & \left[-\sin\delta\rho^0 + \cos\delta\left(\sin\theta_V\omega_8 + \cos\theta_V\omega_0\right)\right](\cos\theta_P\eta_8 - \sin\theta_P\eta_0) \\
&= -\sin\delta\cos\theta_P \cdot \rho^0\eta_8 + \sin\delta\sin\theta_P \cdot \rho^0\eta_0 + \cos\delta\sin\theta_V\cos\theta_P \cdot \omega_8\eta_8 \\
&\quad - \cos\delta\sin\theta_V\sin\theta_P \cdot \omega_8\eta_0 + \cos\delta\cos\theta_V\cos\theta_P \cdot \omega_0\eta_8 - \cos\delta\cos\theta_V\sin\theta_P \cdot \omega_0\eta_0
\end{aligned}$$

Note: only ρ^0 could furnish non-zero result for so-called "VPP" vertex

$$\begin{aligned}
\mathcal{L}_{(2)}^{\mathrm{V}} &= \frac{F_V}{2\sqrt{2}}\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle \supset \frac{iG_V}{\sqrt{2}}\langle V_{\mu\nu}u^\mu u^\nu\rangle \\
&= \frac{iG_V}{\sqrt{2}}\left\langle \frac{1}{\sqrt{2}}\begin{pmatrix} \rho^0 & . & . \\ . & -\rho^0 & . \\ . & . & 0 \end{pmatrix}_{\mu\nu} \cdot \frac{2}{F^2}\partial^\mu\partial^\nu\begin{pmatrix} \pi^+\pi^- & . & . \\ . & \pi^-\pi^+ & . \\ . & . & 0 \end{pmatrix}\right\rangle \\
&= \frac{2\sqrt{2}}{F^2}(\rho^0)_{\mu\nu}\partial^\mu\pi^+\partial^\nu\pi^-
\end{aligned}$$

so only need to add $\cos\delta$ for ρ propagator or $-\sin\delta$ for ω propagator.

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Note for VJP, we focus on $\{V^{\mu\nu}, f_+^{\rho\alpha}\}\nabla_\alpha u^\sigma$, since such a structure often appears ...

$$
\begin{aligned}
\mathcal{O}^1_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\alpha}\}\nabla_\alpha u^\sigma\rangle \\
&\approx \frac{4\sqrt{2}e}{3F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\left\langle \text{diag}\begin{pmatrix} \frac{2\rho^0}{\sqrt{2}}+\frac{2\omega_8}{\sqrt{6}}+\frac{2\omega_0}{\sqrt{3}} \\ \frac{\rho^0}{\sqrt{2}}-\frac{\omega_8}{\sqrt{6}}-\frac{\omega_0}{\sqrt{3}} \\ \frac{2\omega_8}{\sqrt{6}}-\frac{\omega_0}{\sqrt{3}}\end{pmatrix}^{\mu\nu}\cdot\partial_\alpha\partial^\sigma\text{diag}\begin{pmatrix}\frac{\eta_8}{\sqrt{6}}+\frac{\eta_0}{\sqrt{3}} \\ \frac{\eta_8}{\sqrt{6}}+\frac{\eta_0}{\sqrt{3}} \\ \frac{\eta_0}{\sqrt{3}}-2\frac{\eta_8}{\sqrt{6}}\end{pmatrix}\right\rangle \\
&= \frac{4\sqrt{2}e}{3F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\Big(\rho^0\eta_8/\sqrt{3}+2\rho^0\eta_0/\sqrt{6}+\omega_8\eta_8/3+\sqrt{2}\omega_8\eta_0/3+\sqrt{2}\omega_0\eta_8/3+2\omega_0\eta_0/3 \\
&+\rho^0\eta_8/2\sqrt{3}+\rho^0\eta_0/\sqrt{6}-\omega_8\eta_8/6-\omega_8\eta_0/3\sqrt{2}-\omega_0\eta_8/3\sqrt{2}-\omega_0\eta_0/3 \\
&+\sqrt{2}\omega_8\eta_0/3-2\omega_8\eta_8/3-\omega_0\eta_0/3+\sqrt{2}\omega_0\eta_8/3\Big)^{\mu\nu}\cdots\partial_\alpha\partial^\sigma \\
&= \frac{4\sqrt{2}e}{3F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\left(\frac{\sqrt{3}}{2}(\rho^0)^{\mu\nu}\partial_\alpha\partial^\sigma\eta_8+\frac{\sqrt{6}}{2}\rho^0\eta_0-\frac{1}{2}\omega_8\eta_8+\frac{\sqrt{2}}{2}\omega_8\eta_0+\frac{\sqrt{2}}{2}\omega_0\eta_8\right)
\end{aligned}
$$

neglect some $\mu\nu$ and $\partial_\alpha\partial^\sigma$ for convenience.

$$
\begin{aligned}
\mathcal{O}^2_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\alpha}, f_+^{\rho\sigma}\}\nabla_\alpha u^\nu\rangle \\
&= \frac{4\sqrt{2}e}{3F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\left(\frac{\sqrt{3}}{2}(\rho^0)^{\mu\alpha}\partial_\alpha\partial^\nu\eta_8+\frac{\sqrt{6}}{2}\rho^0\eta_0-\frac{1}{2}\omega_8\eta_8+\frac{\sqrt{2}}{2}\omega_8\eta_0+\frac{\sqrt{2}}{2}\omega_0\eta_8\right) \\
\mathcal{O}^5_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu}, f_+^{\rho\alpha}\}u^\sigma\rangle \\
&= \frac{4\sqrt{2}e}{3F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\alpha - \partial^\alpha\mathscr{A}^\rho)\left(\frac{\sqrt{3}}{2}\partial_\alpha(\rho^0)^{\mu\nu}\partial^\sigma\eta_8+\frac{\sqrt{6}}{2}\rho^0\eta_0-\frac{1}{2}\omega_8\eta_8+\frac{\sqrt{2}}{2}\omega_8\eta_0+\frac{\sqrt{2}}{2}\omega_0\eta_8\right) \\
\mathcal{O}^6_{\rm VJP} &= \epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\alpha}, f_+^{\rho\sigma}\}u^\nu\rangle \\
&= \frac{4\sqrt{2}e}{3F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\left(\frac{\sqrt{3}}{2}\partial_\alpha(\rho^0)^{\mu\nu}\partial^\nu\eta_8+\frac{\sqrt{6}}{2}\rho^0\eta_0-\frac{1}{2}\omega_8\eta_8+\frac{\sqrt{2}}{2}\omega_8\eta_0+\frac{\sqrt{2}}{2}\omega_0\eta_8\right)
\end{aligned}
$$

Insert $\rho\eta$ mixing angles into the big parentheses for ρ propagator:

$$
\begin{aligned}
&\left(\frac{\sqrt{3}}{2}(\rho^0)\eta_8+\frac{\sqrt{6}}{2}\rho^0\eta_0-\frac{1}{2}\omega_8\eta_8+\frac{\sqrt{2}}{2}\omega_8\eta_0+\frac{\sqrt{2}}{2}\omega_0\eta_8\right) \\
\to&\frac{\sqrt{3}}{2}\cos\delta\cos\theta_P-\frac{\sqrt{6}}{2}\cos\delta\sin\theta_P-\frac{1}{2}\sin\delta\sin\theta_V\cos\theta_P-\frac{\sqrt{2}}{2}\sin\delta\sin\theta_V\sin\theta_P+\frac{\sqrt{2}}{2}\sin\delta\cos\theta_V\cos\theta_P \\
=&\left\{\sqrt{3}\cos\delta(\cos\theta_P-\sqrt{2}\sin\theta_P)+\sin\delta\left[\sqrt{2}\cos\theta_V\cos\theta_P-\sqrt{2}\sin\theta_V\sin\theta_P-\sin\theta_V\cos\theta_P\right]\right\}/2 \\
=&\left\{\sqrt{3}\cos\delta(\cos\theta_P-\sqrt{2}\sin\theta_P)+\sin\delta\left[\sqrt{2}\cos\theta_V\cos\theta_P-\sin\theta_V(\cos\theta_P+\sqrt{2}\sin\theta_P)\right]\right\}/2
\end{aligned}
$$

Similarly, for ω as propagator, insert mixing angles by replacing $\cos\delta$ to $-\sin\delta$, and $\sin\delta$ to $\cos\delta$.

$$\left(\frac{\sqrt{3}}{2}(\rho^0)\eta_8 + \frac{\sqrt{6}}{2}\rho^0\eta_0 - \frac{1}{2}\omega_8\eta_8 + \frac{\sqrt{2}}{2}\omega_8\eta_0 + \frac{\sqrt{2}}{2}\omega_0\eta_8\right)$$
$$\to \left\{-\sqrt{3}\sin\delta(\cos\theta_P - \sqrt{2}\sin\theta_P) + \cos\delta\left[\sqrt{2}\cos\theta_V\cos\theta_P - \sin\theta_V(\cos\theta_P + \sqrt{2}\sin\theta_P)\right]\right\}/2$$

Review the case w/o mixing angles

$$F_c^\eta = \frac{-4\sqrt{6}G_V}{3M_V F^3(M_\rho^2 - s)}\left[c_1(Q^2 + m_\eta^2 - s) - c_2(Q^2 - m_\eta^2 - s) + 8c_3 m_\pi^2 + c_5(Q^2 - m_\eta^2 + s) - 2c_6 s\right]$$

we follow the substitutions below

$$\frac{1}{M_\rho^2 - s} \to BW_R[\eta, \rho, s] \qquad \frac{1}{M_\omega^2 - s} \to BW_R[\eta, \omega, s]$$

and introduce

$$C_{R\eta 1}(Q^2, x, m^2) = (c_1 - c_2 + c_5)Q^2 - (c_1 - c_2 - c_5 + 2c_6)x + (c_1 + c_2 - c_5)m^2$$

By comparing with ideal mixing angles, we re-find the coefficient: (Note two propagators)

$$\begin{aligned}
F_{c1256}^\eta = & \frac{-4\sqrt{2}G_V}{3M_V F^3}\cos\delta\left\{\sqrt{3}\cos\delta(\cos\theta_P - \sqrt{2}\sin\theta_P) + \sin\delta\left[\sqrt{2}\cos\theta_V\cos\theta_P\right.\right. \\
& \left.\left. - \sin\theta_V(\cos\theta_P + \sqrt{2}\sin\theta_P)\right]\right\}\cdot BW_R[\eta, \rho, s]\, C_{R\eta 1}(Q^2, s, m_\eta^2) \\
& + \frac{-4\sqrt{2}G_V}{3M_V F^3}\sin\delta\left\{-\sqrt{3}\sin\delta(\cos\theta_P - \sqrt{2}\sin\theta_P) + \cos\delta\left[\sqrt{2}\cos\theta_V\cos\theta_P\right.\right. \\
& \left.\left. - \sin\theta_V(\cos\theta_P + \sqrt{2}\sin\theta_P)\right]\right\}\cdot BW_R[\eta, \omega, s]\, C_{R\eta 1}(Q^2, s, m_\eta^2)
\end{aligned}$$

16.3.2 VJP 3

$$\begin{aligned}
\mathcal{O}_{\rm VJP}^3 = & i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, f_+^{\rho\sigma}\}\chi_-\rangle = i\epsilon_{\mu\nu\rho\sigma}\left\langle\{V^{\mu\nu}, f_+^{\rho\sigma}\}\cdot(-\frac{2\sqrt{2}i}{F})\phi\chi\right\rangle \\
= & \frac{-8\sqrt{2}e}{3F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\left\langle\begin{pmatrix}\frac{2\rho^0}{\sqrt{2}} + \frac{2\omega_8}{\sqrt{6}} + \frac{2\omega_0}{\sqrt{3}} & & \\ & \frac{\rho^0}{\sqrt{2}} - \frac{\omega_8}{\sqrt{6}} - \frac{\omega_0}{\sqrt{3}} & \\ & & \frac{2\omega_8}{\sqrt{6}} - \frac{\omega_0}{\sqrt{3}}\end{pmatrix}^{\mu\nu}\cdot\begin{pmatrix}\frac{\eta_8}{\sqrt{6}} + \frac{\eta_0}{\sqrt{3}} & & \\ & \frac{\eta_8}{\sqrt{6}} + \frac{\eta_0}{\sqrt{3}} & \\ & & \frac{\eta_0}{\sqrt{3}} - 2\frac{\eta_8}{\sqrt{6}}\end{pmatrix}\cdot\begin{pmatrix}m_\pi^2 & & \\ & m_\pi^2 & \\ & & 2m_K^2 - m_\pi^2\end{pmatrix}\right\rangle \\
= & \frac{-8\sqrt{2}e}{3F}\epsilon_{\mu\nu\rho\sigma}(\partial^\rho\mathscr{A}^\sigma - \partial^\sigma\mathscr{A}^\rho)\left[m_\pi^2\left(\frac{\sqrt{3}}{2}\rho^0\eta_8 + \frac{\sqrt{6}}{2}\rho^0\eta_0 + \frac{5}{6}\omega_8\eta_8 - \frac{\sqrt{2}}{6}\omega_8\eta_0 - \frac{\sqrt{2}}{6}\omega_0\eta_8 + \frac{2}{3}\omega_0\eta_0\right)\right. \\
& \left. + m_K^2\left(\frac{2\sqrt{2}}{3}\omega_8\eta_0 - \frac{4}{3}\omega_8\eta_8 - \frac{2}{3}\omega_0\eta_0 + \frac{2\sqrt{2}}{3}\omega_0\eta_8\right)\right]
\end{aligned}$$

Note that m_K^2 is taken into consideration here.

Then, insert mixing angles again for ρ propagator

$$
\begin{aligned}
&\left[m_\pi^2(\cdots)+m_K^2(\cdots)\right]\\
\to& m_\pi^2\left(\frac{\sqrt{3}}{2}\cos\delta\cos\theta_P-\frac{\sqrt{6}}{2}\cos\delta\sin\theta_P+\frac{5}{6}\sin\delta\sin\theta_V\cos\theta_P+\frac{\sqrt{2}}{6}\sin\delta\sin\theta_V\sin\theta_P\right.\\
&\qquad\left.-\frac{\sqrt{2}}{6}\sin\delta\cos\theta_V\cos\theta_P-\frac{2}{3}\sin\delta\cos\theta_V\sin\theta_P\right)\\
&+m_K^2\left(-\frac{2\sqrt{2}}{3}\sin\delta\sin\theta_V\sin\theta_P-\frac{4}{3}\sin\delta\sin\theta_V\cos\theta_P+\frac{2}{3}\sin\delta\cos\theta_V\sin\theta_P+\frac{2\sqrt{2}}{3}\sin\delta\cos\theta_V\cos\theta_P\right)\\
=&\frac{m_\pi^2}{6}\left[3\sqrt{3}\cos\delta(\cos\theta_P-\sqrt{2}\sin\theta_P)+\sin\delta\left(5\sin\theta_V\cos\theta_P-4\cos\theta_V\sin\theta_P-\sqrt{2}\cos(\theta_V+\theta_P)\right)\right]\\
&\qquad+\frac{m_K^2}{6}\cdot 4\sin\delta\left[\sqrt{2}\cos(\theta_V+\theta_P)-2\sin\theta_V\cos\theta_P+\cos\theta_V\sin\theta_P\right]\\
&=const(A)/6
\end{aligned}
$$

Similarly, replace $\cos\delta$ by $-\sin\delta$ and $\sin\delta$ by $\cos\theta$ for ω propagator, and suppose the result as $const(B)/6$. Introduce $C_{R\eta 2}=8c_3$, we can write

$$
F_{c3}^\eta=\frac{-4\sqrt{2}G_V}{9M_V F^3}\left\{\cos\delta\cdot const(A)\cdot BW_R[\eta,\rho,s]-\sin\delta\cdot const(B)\cdot BW_R[\eta,\omega,s]\right\}C_{R\eta 2}
$$

16.4 VVP, F_d^η

$$
\begin{aligned}
\mathcal{O}^1_{\mathrm{VVP}}&=\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu},V^{\rho\alpha}\}\nabla_\alpha u^\sigma\rangle\\
&=\frac{-2\sqrt{2}}{F}\epsilon_{\mu\nu\rho\sigma}\left\langle\begin{pmatrix}\frac{\rho^0}{\sqrt2}+\frac{\omega_8}{\sqrt6}+\frac{\omega_0}{\sqrt3}\\ -\frac{\rho^0}{\sqrt2}+\frac{\omega_8}{\sqrt6}+\frac{\omega_0}{\sqrt3}\\ -\frac{2\omega_8}{\sqrt6}+\frac{\omega_0}{\sqrt3}\end{pmatrix}^{\mu\nu}_{\mathrm{diag}}\cdot\begin{pmatrix}\frac{\rho^0}{\sqrt2}+\frac{\omega_8}{\sqrt6}+\frac{\omega_0}{\sqrt3}\\ -\frac{\rho^0}{\sqrt2}+\frac{\omega_8}{\sqrt6}+\frac{\omega_0}{\sqrt3}\\ -\frac{2\omega_8}{\sqrt6}+\frac{\omega_0}{\sqrt3}\end{pmatrix}^{\rho\alpha}_{\mathrm{diag}}\cdot\partial_\alpha\partial^\sigma\begin{pmatrix}\frac{\eta_8}{\sqrt6}+\frac{\eta_0}{\sqrt3}\\ \frac{\eta_8}{\sqrt6}+\frac{\eta_0}{\sqrt3}\\ \frac{\eta_0}{\sqrt3}-2\frac{\eta_8}{\sqrt6}\end{pmatrix}_{\mathrm{diag}}\right\rangle\\
&=\frac{-2\sqrt{2}}{F}\epsilon_{\mu\nu\rho\sigma}\times\text{coefficients}
\end{aligned}
$$

with the corresponding coefficients listed below. (left for η_8, right for η_0)

$2\backslash 1$	ρ^0	ω_8	ω_0
ρ^0	$\frac{\sqrt2}{\sqrt3}$	0	0
ω_8	0	$-\frac{\sqrt2}{\sqrt3}$	$\frac{4}{\sqrt3}$
ω_0	0	$\frac{4}{\sqrt3}$	0

$2\backslash 1$	ρ^0	ω_8	ω_0
ρ^0	$\frac{2}{\sqrt3}$	0	0
ω_8	0	$\frac{2}{\sqrt3}$	0
ω_0	0	0	$\frac{2}{\sqrt3}$

Also, there would be the same coefficients for $\mathcal{O}^3_{\mathrm{VVP}}=\epsilon_{\mu\nu\rho\sigma}\langle\{\nabla_\alpha V^{\mu\nu},V^{\rho\alpha}\}u^\sigma\rangle$.

However, mass terms would be introduced for $\mathcal{O}^2_{\mathrm{VVP}}=i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu},V^{\rho\sigma}\}\chi_-\rangle$

$$
\begin{aligned}
\mathcal{O}^2_{\mathrm{VVP}} &= i\epsilon_{\mu\nu\rho\sigma}\langle\{V^{\mu\nu}, V^{\rho\sigma}\}\,\chi_-\rangle \\
&= \frac{-4\sqrt{2}i}{F}\epsilon_{\mu\nu\rho\sigma}\left\langle \begin{pmatrix} \frac{\rho^0}{\sqrt{2}}+\frac{\omega_8}{\sqrt{6}}+\frac{\omega_0}{\sqrt{3}} \\ -\frac{\rho^0}{\sqrt{2}}+\frac{\omega_8}{\sqrt{6}}+\frac{\omega_0}{\sqrt{3}} \\ -\frac{2\omega_8}{\sqrt{6}}+\frac{\omega_0}{\sqrt{3}} \end{pmatrix}^{\mu\nu}_{\mathrm{diag}} \cdot \begin{pmatrix} \frac{\rho^0}{\sqrt{2}}+\frac{\omega_8}{\sqrt{6}}+\frac{\omega_0}{\sqrt{3}} \\ -\frac{\rho^0}{\sqrt{2}}+\frac{\omega_8}{\sqrt{6}}+\frac{\omega_0}{\sqrt{3}} \\ -\frac{2\omega_8}{\sqrt{6}}+\frac{\omega_0}{\sqrt{3}} \end{pmatrix}^{\rho\sigma}_{\mathrm{diag}} \cdot \right. \\
&\left. \begin{pmatrix} \frac{\eta_8}{\sqrt{6}}+\frac{\eta_0}{\sqrt{3}} \\ \frac{\eta_8}{\sqrt{6}}+\frac{\eta_0}{\sqrt{3}} \\ \frac{\eta_0}{\sqrt{3}}-2\frac{\eta_8}{\sqrt{6}} \end{pmatrix}_{\mathrm{diag}} \cdot \begin{pmatrix} M_\pi^2 & 0 & 0 \\ 0 & M_\pi^2 & 0 \\ 0 & 0 & 2M_K^2-M_\pi^2 \end{pmatrix} \right\rangle \\
&= \frac{-2\sqrt{2}}{F}\epsilon_{\mu\nu\rho\sigma} \times \text{coefficients}
\end{aligned}
$$

with the corresponding coefficients listed below. (left for η_8, right for η_0)

$2\backslash 1$	ρ^0	ω_8	ω_0
ρ^0	$\frac{\sqrt{2}}{\sqrt{3}}\mathrm{M}_\pi^2$	0	0
ω_8	0	$\frac{\sqrt{2}}{3\sqrt{3}}(5M_\pi^2-8M_K^2)$	$\frac{4}{3\sqrt{3}}(4M_K^2-M_\pi^2)$
ω_0	0	$\frac{4}{3\sqrt{3}}(4M_K^2-M_\pi^2)$	$\frac{4\sqrt{2}}{3\sqrt{3}}(M_\pi^2-M_K^2)$

$2\backslash 1$	ρ^0	ω_8	ω_0
ρ^0	$\frac{2}{\sqrt{3}}\mathrm{M}_\pi^2$	0	0
ω_8	0	$\frac{2}{3\sqrt{3}}(4M_K^2-M_\pi^2)$	$\frac{8\sqrt{2}}{3\sqrt{3}}(M_\pi^2-M_K^2)$
ω_0	0	$\frac{8\sqrt{2}}{3\sqrt{3}}(M_\pi^2-M_K^2)$	$\frac{2}{3\sqrt{3}}(M_\pi^2+2M_K^2)$

Finally, with the help of *Mathematica*, we can derive complet form factors with mixing angles.